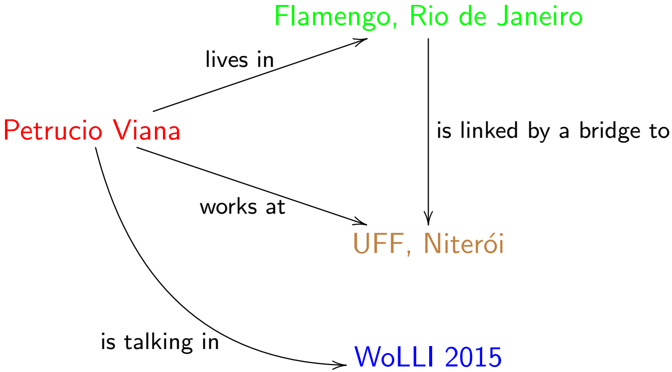
Methods of proof for residuated algebras of binary relations



Joint work with Marcia Cerioli (COPPE/IM, UFRJ)

Outline

1. Binary relations and some of their operations
2. Residuated algebras of binary relations
3. Algebraic and quasi-algebraic theories of residuated algebras of binary relations
4. Calculational reasoning
5. Diagrammatic reasoning
6. Perspectives

1. Binary relations and some of their operations

Binary relations

Let U be a set.

Elements of U are usually denoted by $u, v, w, \ldots$

A *binary relation on* U is a subset of $U \times U$.

$\mathrm{2Rel}U$ is the set of all binary relations on U.

Elements of $\mathrm{2Rel}U$ are usually denoted by $R, S, T, \ldots$

Operations on binary relations

Let $R, S \in 2\mathrm{Rel}U$.

Booleans

The *union* of R and S is:

$$R \cup S = \{(u, v) \in U : (u, v) \in R \text{ or } (u, v) \in S\}$$

The *intersection* of R and S is:

$$R \cup S = \{(u, v) \in U : (u, v) \in R \text{ and } (u, v) \in S\}$$

Operations on binary relations

Let $R, S \in 2\mathrm{Rel}U$.

Peirceans

The *composition* of R and S is:

$$R \circ S = \{(u, v) \in U : \exists w \in U[(u, w) \in R \text{ and } (w, v) \in S]\}$$

The *reversion* of R is:

$$R^{-1} = \{(u, v) \in U : (v, u) \in R\}$$

Operations on binary relations

Let $R, S \in 2\mathrm{Rel}U$.

Between Booleans and Peirceans

The *left residuation* of R and S is:

$$R \backslash S = \{(u, v) \in U : \forall w \in U[\text{ if } (w, u) \in R, \text{ then } (w, v) \in S]\}$$

The *right residuation* of R and S is:

$$R / S = \{(u, v) \in U : \forall w \in U[\text{ if } (v, w) \in S, \text{ then } (u, w) \in R]\}$$

Motivations for residuals

- Algebra: M. Ward and R.P. Dilworth. *Residuated lattices*. *Trans. Amer. Math. Soc.* **45**: 335–54 (1939)

- Computer Science: C.A.R Hoare and H. Jifeng. The weakest prespecification. *Fund. Inform.* **9**: Part I 51–84, Part II 217–252 (1986)

- Linguistics: J. Lambek. The mathematics of sentence structure. *Amer. Math. Monthly* **65**: 154–170 (1958)

- Logic: N. Galatos, P. Jipsen, T. Kowalski, and H. Ono . *Residuated Lattices. An Algebraic Glimpse at Substructural Logics*. Elsevier (2007)

2. Residuated algebras of binary relations

Residuated algebras of relations

Let U be a set.

Let $A \subseteq 2\mathrm{Rel}U$ be closed under all the operations $\cup$, $\cap$, $\circ$, ${}^{-1}$, $\backslash$ and $/$.

The *residuated algebra of binary relations* on U based on A is the algebra:

$$\mathfrak{A} = \langle A, \cup, \cap, \circ, {}^{-1}, \backslash, / \rangle$$

$\mathcal{A}2\mathrm{Rel}$ is the class of all residuated algebra of binary relations.

Elements of $\mathcal{A}2\mathrm{Rel}$ are usually denoted by $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$

Residuated algebras of relations

Aka lattice-ordered involuted residuated semigroups:

1. *Lattice*: $R \cup S$ is a supremum and $R \cap S$ is a infimum.
2. *Ordered*: $R \leq S$ (iff $R \cup S = S$ iff $R \cap S = R$) is a parcial ordering.
3. *Semigroup*: $R \circ S$ is a not necessarily commutative multiplication.
4. *Involuted*: $(R^{-1})^{-1} = R$ and $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$.
5. *Residuated*: $\backslash$ is the left-inverse of $\circ$ and $/$ is the right inverse of $\circ$.

3. Algebraic and quasi-algebraic theories of residuated algebras of binary relations

Terms and inclusions

The **terms**, typically denoted by R, S, T, ..., are generated by:

$$R ::= X \mid R \cup R \mid R \cap S \mid R \circ R \mid R \backslash R \mid R/R \mid R^{-1}$$

where $X \in \mathsf{Var}$, a set of *variables* fixed in advance.

A **quasi-equality** is an expression of the form

$$R \subseteq S$$

where R and S ate terms.

Horn quasi-equalities

A **Horn quasi-equality** is an expression of the form

$$R_1 \subseteq S_1, \ldots, R_n \subseteq S_n \Rightarrow R \subseteq S$$

where $R_1, S_2, \ldots, R_n, S_n, R, S$ are terms.

Valuations and values

Let $\mathfrak{A} \in \mathcal{A}2\mathrm{Rel}$.

A **valuation** on $\mathfrak{A}$ is a function $v : \mathrm{Var} \to A$.

Let R be a term, $\mathfrak{A} \in \mathcal{A}2\mathrm{Rel}$, and v be a valuation on $\mathfrak{A}$.
The **value** of R in $\mathfrak{A}$ according to v, denoted by $R_v^{\mathfrak{A}}$ is defined by:

$$
\begin{array}{lcl}
X_v^{\mathfrak{A}} & = & vX \\
(R \cup S)_v^{\mathfrak{A}} & = & R_v^{\mathfrak{A}} \cup S_v^{\mathfrak{A}} \\
(R \cap S)_v^{\mathfrak{A}} & = & R_v^{\mathfrak{A}} \cap S_v^{\mathfrak{A}} \\
(R \circ S)_v^{\mathfrak{A}} & = & R_v^{\mathfrak{A}} \circ S_v^{\mathfrak{A}} \\
(R \backslash S)_v^{\mathfrak{A}} & = & R_v^{\mathfrak{A}} \backslash S_v^{\mathfrak{A}} \\
(R^{-1})_v^{\mathfrak{A}} & = & (R_v^{\mathfrak{A}})^{-1}
\end{array}
$$

Truth and validity

Let $R \subseteq S$ be a quasi-equality, $\mathfrak{A} \in \mathcal{A}2\mathsf{Rel}$, and v be a valuation on $\mathfrak{A}$.

$R \subseteq S$ is **true** on $\mathfrak{A}$ under v if $R_v^{\mathfrak{A}} \subseteq S_v^{\mathfrak{A}}$.

$R \subseteq S$ is **identically true** on $\mathfrak{A}$, or $\mathfrak{A}$ is a **model** of $R \subseteq S$, if $R \subseteq S$ is true on $\mathfrak{A}$ under v, for every valuation v.

$R \subseteq S$ is **valid** if every residuated algebra of relations $\mathfrak{A}$ is a model of $R \subseteq S$.

Validity and consequence

Let

$$R_1 \subseteq S_1, \ldots, R_n \subseteq S_n \Rightarrow R \subseteq S \tag{1}$$

be a Horn quasi-equality, $\mathfrak{A} \in \mathcal{A}2\text{Rels}$, and v be a valuation on $\mathfrak{A}$.

(1) is **valid**, or $R \subseteq S$ is a **consequence** of $R_1 \subseteq S_1$, ..., $R_n \subseteq S_n$, if every model of all $R_1 \subseteq S_1$, ..., $R_n \subseteq S_n$ is a model of $R \subseteq S$.

From quasi-equalities to equalities and back

An **equality** is an expression of the form

$$R = S$$

where R and S ate terms.

A **Horn equality** is an expression of the form

$$R_1 = S_1, \ldots, R_n = S_n \Rightarrow R = S$$

where $R_1, S_2, \ldots, R_n, S_n, R, S$ are terms.

From quasi-equalities to equalities and back

True, identically true, and valid equalities are defined as usual.

From quasi-equalities to equalities and back

Since

$$R \subseteq S \text{ is valid iff } R \cap S \subseteq S \text{ and } S \subseteq R \cap S \text{ are both valid,}$$

we can consider to build the algebraic and the quasi-algebraic theories of the residuated algebras of relations on the top of the *logic of equality*.

But, taking equational logic as the subjacent logic, we have the following . . .

Negative results

The set of all valid equalities (quasi-equalities) is not finitely axiomatizable (Mikulás, IGPL, 2010).

The set of all valid Horn equalities (Horn quasi-equalities) is not finitely axiomatizable (Andréka and Mikulás, JoLLI, 1994).

Negative results

One proper question is:

are there interesting alternatives for equational reasoning on residuated algebras of binary relations?

4. Calculational reasoning

Quasi-posets

Let P be a set and R be a binary relation on P.

$\langle P, R \rangle$ is a **quasi-poset** if R is reflexive and transitive (but not necessarily antisymmetric) on P.

Galois connections

Let $\mathfrak{P}_1 = \langle P_1, \leq_1 \rangle$, $\mathfrak{P}_2 = \langle P_2, \leq_2 \rangle$ be quasi-posets, and $f : P_1 \to P_2$, $g : P_2 \to P_1$ be functions.

$\langle \mathfrak{P}_1, \mathfrak{P}_2, f, g \rangle$ is a **Galois connection** if, for all $x \in P_1$ and $y \in P_2$:

$$fx \leq_2 y \iff x \leq_1 gy$$

Calculational rules

Quasi-poset rules

$$\frac{\top}{x \leq x}\,\text{Ref} \qquad \frac{\begin{array}{c} x \leq y \\ \vdots \\ y \leq z \end{array}}{x \leq z}\,\text{Tra}$$

GC rules

$$\frac{fx \leq y}{x \leq gy}\,\text{GC} \qquad \frac{x \leq gy}{fx \leq y}\,\text{GC}$$

These rules aloud us to perform both direct and indirect calculational reasoning (without negation).

Direct calculational proofs

A **direct calculational proof** of $t_1 \leq t_2$ is a sequence

$$\langle t_1 \leq t_2, t_3 \leq t_4, \ldots, t_{n-1} \leq t_n \rangle$$

such that, for each i, $3 \leq i \leq n$, $t_i \leq t_{i+1}$, at least one oh the following conditions hold:

1. $t_i \leq t_{i+1}$ is an axiom.
2. $t_i \leq t_{i+1}$ is obtained from previou(s) quasi-equation(s) in the sequence by one application of some calculational rule.
3. $t_{n-1} \leq t_n$ is an axiom.

Start with $t_1 \leq t_2$ and applying axioms and calculational rules arrive in an axiom.

Direct calculational proofs from hypothesis

Let Γ be a set of quasi-equations.

A **direct calculational proof** of $t_1 \leq t_2$ from Γ is a sequence

$$\langle t_1 \leq t_2, t_3 \leq t_4, \ldots, t_{n-1} \leq t_n \rangle$$

such that, for each $t_i \leq t_{i+1}$, where $3 \leq i \leq n$, at least one of the following conditions hold:

1. $t_i \leq t_{i+1}$ is an axiom
2. $t_i \leq t_{i+1} \in \Gamma$
3. $t_i \leq t_{i+1}$ is obtained from previou(s) quasi-equation(s) in the sequence by one application of some Calculational Rule.
4. $t_{n-1} \leq t_n$ is an axiom or belongs to Γ.

Start with $t_1 \leq t_2$ and applying axioms, hyphotesis, and calculational rules arrive in an axiom or hyphotesis.

$\cup$ defines a Galois connection

Let $\langle \mathfrak{A}, \subseteq \rangle \in \mathcal{A}2\text{Rel}$ and take $\langle \mathfrak{A} \times \mathfrak{A}, \subseteq \times \subseteq \rangle \in \mathcal{A}2\text{Rel}$.

For all $X, Y \in A$, we define $f : A \times A \to A$ by:

$$f(X, Y) = X \cup Y$$

and $g : A \to A \times A$ by:

$$g(X) = (X, X)$$

With these notations, for all $R, S, T \in A$:

$$R \cup S \subseteq T \iff R \subseteq T \text{ and } S \subseteq T$$

is the same as

$$f(R, S) \subseteq T \Leftrightarrow (R, S) \subseteq g(T)$$

$\backslash$ defines a family of Galois connections

Let $\langle \mathfrak{A}, \subseteq \rangle \in \mathcal{A}$2Rel.

For every $R \in A$, we define:

$$f_R(X) = R \circ X$$

and

$$g_R(X) = R \backslash X$$

With these notations, we have that

$$R \circ S \subseteq T \Leftrightarrow S \subseteq R \backslash T$$

is the same as

$$f_R(S) \subseteq T \Leftrightarrow S \subseteq g_R(T)$$

$\cap$, $^{-1}$ and $/$ define Galois connections

Sorry, no time to enter in details!

Basic arithmetical results

T_1) $S \subseteq R \backslash (R \circ S)$

$$S \subseteq R \backslash (R \circ S)$$
$$\Updownarrow \text{ GC}$$
$$R \circ S \subseteq R \circ S$$
$$\Updownarrow \text{ Ref}$$
$$\top$$

Basic arithmetical results

T_2) $R \circ (R \backslash S) \subseteq S$

$$R \circ (R \backslash S) \subseteq S$$
$$\Updownarrow \text{ GC}$$
$$R \backslash S \subseteq R \backslash S$$
$$\Updownarrow \text{ Ref}$$
$$\top$$

Basic arithmetical results

T_3) $R\backslash(S \cap T) \subseteq (R\backslash S) \cap (R\backslash T)$

$$R\backslash(S \cap T)] \subseteq (R\backslash S) \cap (R\backslash T)$$
$$\Updownarrow \text{ GC}$$
$$R\backslash(S \cap T)] \subseteq R\backslash S \wedge R\backslash(S \cap T) \subseteq S\backslash T$$
$$\Updownarrow \text{ GC}$$
$$R \circ [R\backslash(S \cap T)] \subseteq S \wedge R \circ [R\backslash(S \cap T)] \subseteq T$$
$$\Updownarrow \text{ GC}$$
$$R \circ [R\backslash(S \cap T)] \subseteq S \cap T$$
$$\Updownarrow \text{ GC}$$
$$R\backslash(S \cap T) \subseteq R \circ (S \cap T)$$
$$\Updownarrow \text{ Ref}$$
$$\top$$

Basic arithmetical results

$T_4)\ S \subseteq T \Longrightarrow R\backslash S \subseteq R\backslash T$

$$
\begin{array}{l}
S \subseteq T \\
\Updownarrow \ T_2 \\
R \circ (R\backslash S) \subseteq T \\
\Updownarrow \ \text{GC} \\
R\backslash S \subseteq R\backslash T
\end{array}
$$

By T_2, $R \circ (R\backslash S) \subseteq S$.

Basic arithmetical results

T_5) $T_1, T_2, T_3 \Longrightarrow$ GC for $\backslash$

$$
\begin{array}{l}
R \circ S \subseteq T \\
\Downarrow \text{Mon, Ide} \\
R \circ S \subseteq (R \circ S) \cap T \\
\Downarrow T_4 \\
R \backslash (R \circ S) \subseteq R \backslash [(R \circ S) \cap T] \\
\Downarrow T_1 \\
S \subseteq R \backslash [(R \circ S) \cap T] \\
\Downarrow T_3 \\
S \subseteq R \backslash T
\end{array}
$$

By T_1, $S \subseteq R \backslash (R \circ S)$.
By T_3, $R \backslash [(R \circ S) \cap T] \subseteq R \backslash T$.

Basic arithmetical results

T_5) $T_1, T_2, T_3 \Longrightarrow$ GC for $\backslash$

$$
\begin{array}{l}
S \subseteq R\backslash T \\
\Downarrow \text{ Mon} \\
R \circ S \subseteq R \circ (R\backslash T) \\
\Downarrow T_2 \\
R \circ S \subseteq T
\end{array}
$$

By T_2, $R \circ (R\backslash S) \subseteq S$

Indirect calculational proofs

An **indirect calculational proof** of $t_1 \leq t_n$ is a sequence

$$\langle x \leq t_1, t_2 \leq t_3, \ldots, x \leq t_n \rangle$$

such that $t_i \leq t_{i+1}$ —for each i, $2 \leq i \leq n-1$— and $x \leq t_n$ are obtained from previou(s) quasi-equation(s) in the sequence by one application of some calculational rule.

Suppose $x \leq t_1$ and prove $x \leq t_2$ by applying the calculational rules.

Indirect calculational proofs from hyphotesis

Let Γ be a set of quasi-equations.

A **direct calculational proof** of $t_1 \leq t_n$ from Γ is a sequence

$$\langle x \leq t_1, t_2 \leq t_3, \ldots, x \leq t_n \rangle$$

such that, for each $t_i \leq t_{i+1}$, where $2 \leq i \leq n-1$, at least one of the following conditions hold:

1. $t_i \leq t_{i+1}$ is an axiom
2. $t_i \leq t_{i+1} \in \Gamma$
3. $t_i \leq t_{i+1}$ is obtained from previou(s) quasi-equation(s) in the sequence by one application of some calculational rule.
4. $x \leq t_n$ is an axiom or belongs to Γ.

Suppose $x \leq t_1$ and prove $x \leq t_2$ by applying axioms, hyphotesis, and calculational rules.

Basic arithmetical results

T_6) $(R\backslash S) \cap (R\backslash T) \subseteq R\backslash(S \cap T)$

$$
\begin{array}{l}
X \subseteq (R\backslash S) \cap (R\backslash T) \\
\Updownarrow \text{ GC} \\
X \subseteq R\backslash S \wedge X \subseteq R\backslash T \\
\Updownarrow \text{ GC} \\
R \circ X \subseteq S \wedge R \circ X \subseteq T \\
\Updownarrow \text{ GC} \\
R \circ X \subseteq S \cap T \\
\Updownarrow \text{ GC} \\
X \subseteq R\backslash(S \cap T)
\end{array}
$$

Hence, $(R\backslash S) \cap (R\backslash T) \subseteq R\backslash(S \cap T)$ and $R\backslash(S \cap T) \subseteq (R\backslash S) \cap (R\backslash T)$ (this is a bonus!).

Some questions

To determine the strengths of:

(1) direct calculational proofs,

(2) direct calculational proofs from hypothesis,

(3) indirect calculational proofs, and

(4) indirect calculational proofs from hyphotesis.

5. Diagrammatic reasoning

Digraphs

A **directed labelled multi graph** is a structure $\langle N, A \rangle$, where:

1. N is a set of **nodes**
2. $A \subseteq N \times \text{Terms} \times N$ is a set of **arcs labeled by terms**

Nodes are usually denoted by $u, v, w, \ldots$

Digraphs are usualy denoted by $\mathfrak{G}, \mathfrak{H}, \mathfrak{I}, \ldots$

Homomorphisms

Let $\mathfrak{G}_1 = \langle N_1, A_1 \rangle$ and $\mathfrak{G}_2 = \langle N_2, A_2 \rangle$ be digraphs.

A **homomorphism from** $\mathfrak{G}_1$ **to** $\mathfrak{G}_2$ is a mapping $h : N_1 \to N_2$ such that:

$$(hu, t, hv) \in A_2 \text{ whenever } (u, t, v) \in A_1$$

A mapping that preserves labels.

2-pointed graphs

A 2-**pointed digraph** is a structure

$$\langle N, A, s, t \rangle,$$

where:

1. $\langle N, A \rangle$ is the **subjacent digraph**
2. $s, t \in N$, where s is the **source** and t is the **target**

2-pointed digraphs are usually denoted by $\langle \mathfrak{G}, s, t \rangle$.

2-pointed Homomorphisms

Let $\mathfrak{G}_1 = \langle N_1, A_1, s_1, t_1 \rangle$ and $\mathfrak{G}_2 = \langle N_2, A_2, s_2, t_2 \rangle$ be 2-pointed digraphs.

A 2-**pointed homomorphism from** $\mathfrak{G}_1$ **to** $\mathfrak{G}_2$ is a homomorphism $h : N_1 \to N_2$ such that:

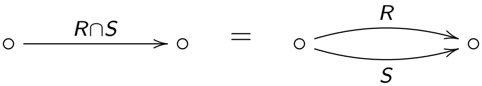
$$hs_1 = s_2 \text{ and } ht_1 = t_2$$

A homomorphism that preserves source and target.

Operations on diagrams

Split digraphs

Paralelize arcs



Operations on diagrams

Sequentialize arcs

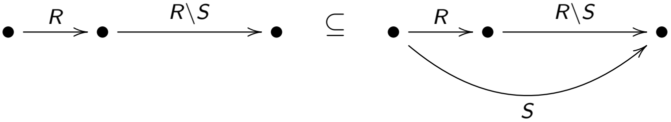
$$\circ \xrightarrow{R \circ S} \circ \quad = \quad \circ \xrightarrow{R} \circ \xrightarrow{S} \circ$$

Revert arcs

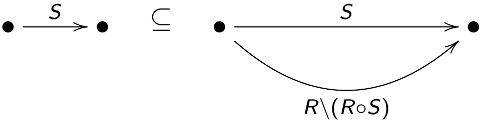
$$\circ \xrightarrow{R^{-1}} \circ \quad = \quad \circ \xleftarrow{R} \circ$$

Operations on diagrams

Close digraphs



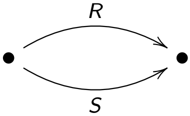
Add residuals



Operations on diagrams

Hyphotesis rule

$$R \subseteq S \quad \wedge \quad \bullet \xrightarrow{R} \bullet \quad \subseteq$$



Hybrid rule

$$S \subseteq T \quad \wedge \quad \bullet \xrightarrow{R\backslash S} \bullet \quad \subseteq \quad \bullet \xrightarrow{R\backslash T} \bullet$$

Basic arithmetical results

Suppose $R \circ S \subseteq T$.

We shall prove $S \subseteq R \backslash T$ by means of diagrams.

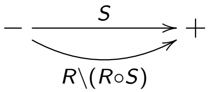
Basic arithmetical results

Start with the graph of the left hand side:

$$- \xrightarrow{S} +$$

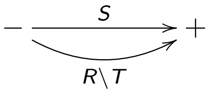
Basic arithmetical results

Apply add residuals:



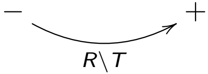
Basic arithmetical results

Apply hybrid rule, together with the hyphotesis $R \circ S \subseteq T$:



Basic arithmetical results

Apply homomorphism, erasing superfluous arcs:



Basic arithmetical results

Suppose $S \subseteq R\backslash T$.

We shall prove $R \circ S \subseteq T$ by means of diagrams.

Basic arithmetical results

Start with the graph of the left hand side:

$$- \xrightarrow{R \circ S} +$$

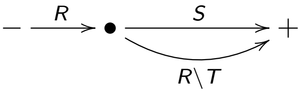
Basic arithmetical results

Apply sequencialize arcs:

$$- \xrightarrow{R} \bullet \xrightarrow{S} +$$

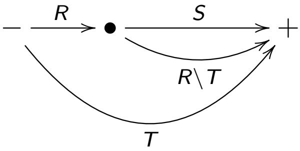
Basic arithmetical results

Apply the hyphotesis $R \subseteq R \backslash T$:



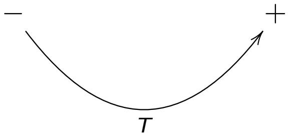
Basic arithmetical results

Apply close diagram:



Basic arithmetical results

Apply homomorphism, arasing superfluous arcs:



Some questions

(1) To determine the strengths of the proofs with graphs.

(2) To compare equational reasoning with calculational reasoning with diagrammatic reasoning.