

By **Graham P. Collins**

**S**tand up and look around. Walk in a circle. Jump in the air. Wave your arms. You are a collection of particles moving about within a small region of a 3-manifold—a three-dimensional space—that extends in all directions for many billions of light-years.

Manifolds are mathematical constructs. The triumph of physics since the time of Galileo and Kepler has been the successful description of reality by mathematics of one flavor or another, such as the mathematics of manifolds. According to physics, everything that happens takes place against the backdrop of three-dimensional space (leaving aside the speculations of string theorists that there are tiny dimensions in addition to the three that are manifest) [see “The Theory Formerly Known as Strings,” by Michael J. Duff; *SCIENTIFIC AMERICAN*, February 1998]. Three dimensions means that three numbers are needed to specify the location of a particle. Near Earth, for instance, the three numbers could be latitude, longitude and altitude.

According to Newtonian physics and traditional quantum physics, the three-dimensional space where everything happens is fixed and immutable. Einstein’s theory of general relativity, in contrast,



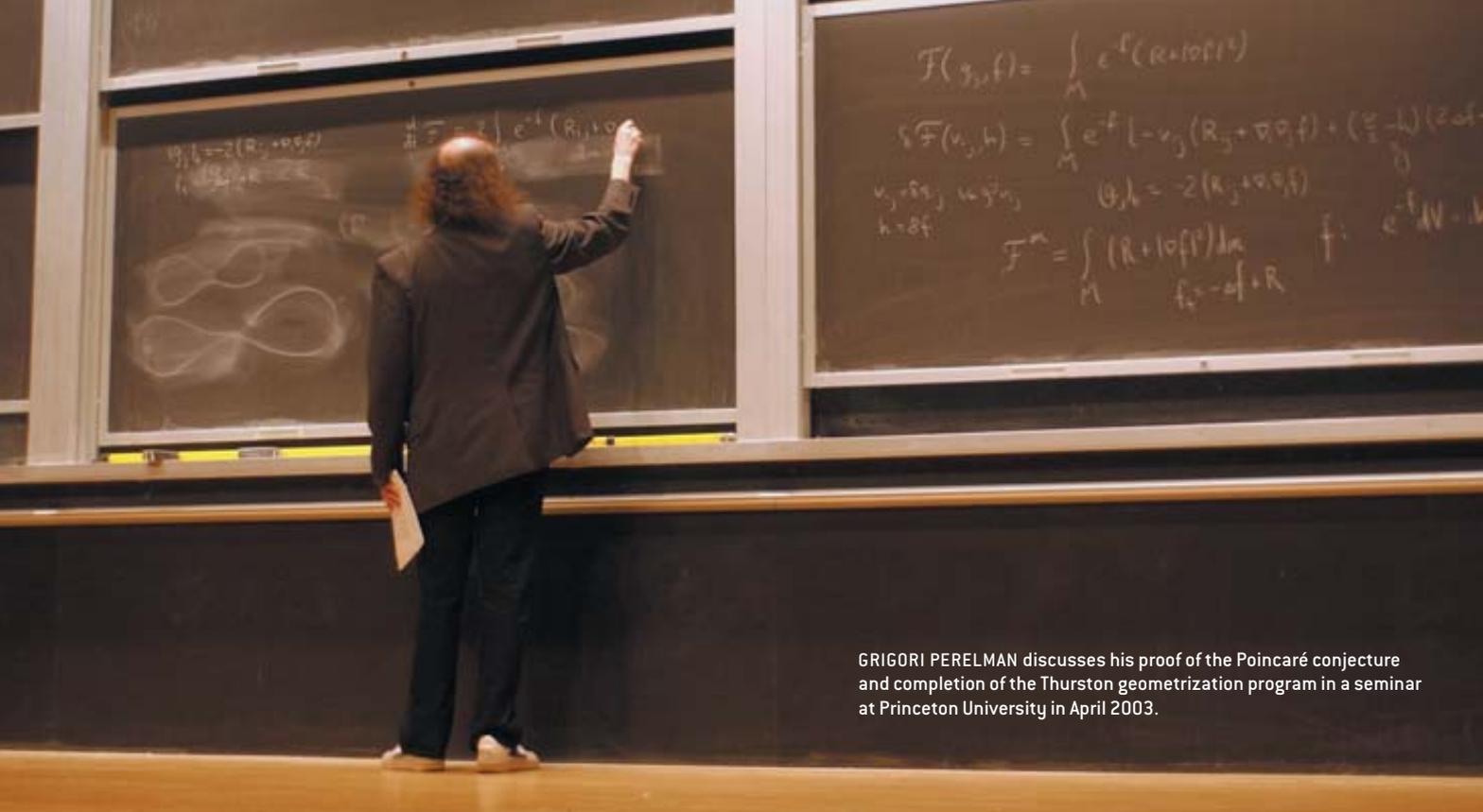
HENRI POINCARÉ conjectured in 1904 that any three-dimensional object that shares certain properties of the three-dimensional sphere can be morphed into a 3-sphere. It took 99 years for mathematicians to prove his conjecture. [Beware: the three-dimensional sphere is probably not what you think it is!]

# The Shapes of Space

A Russian mathematician has proved the century-old Poincaré conjecture and completed the catalogue of three-dimensional spaces. He might earn a \$1-million prize



AIP EMILIO SEGRÈ VISUAL ARCHIVES (photograph); COMPUTER ILLUSTRATION BY JANA BRENNING, WITH 3-D SHAPES BY DON FOLEY (opposite page)



GRIGORI PERELMAN discusses his proof of the Poincaré conjecture and completion of the Thurston geometrization program in a seminar at Princeton University in April 2003.

makes space an active player: the distance from one point to another is influenced by how much matter and energy are nearby and by any gravitational waves that may be passing by [see “Ripples in Space-time,” by W. Wayt Gibbs; SCIENTIFIC AMERICAN, April 2002]. But whether we are dealing with Newtonian or Einsteinian physics and whether space is infinite or finite, space is represented by a 3-manifold. Understanding the properties of 3-manifolds is therefore essential for fully comprehending the foundations on which almost all of physics—and all other sciences—are built. (The 4-manifolds are also important: space and time together form a 4-manifold.)

Mathematicians know a lot about 3-manifolds, yet some of the most basic questions have proved to be the hardest.

The branch of mathematics that studies manifolds is topology. Among the fundamental questions topologists can ask about 3-manifolds are: What is the simplest type of 3-manifold, the one with the least complicated structure? Does it have many cousins that are equally simple, or is it unique? What kinds of 3-manifolds are there?

The answer to the first of those questions has long been known: a space called the 3-sphere is the simplest compact 3-manifold. (Noncompact manifolds can be thought of as being infinite or having an edge. Hereafter I consider only compact manifolds.) The other two questions have been up for grabs for a century but may have been answered in 2002 by Grigori (“Grisha”) Perelman, a Russian mathematician who has most probably

proved a theorem known as the Poincaré conjecture.

First postulated by French mathematician Henri Poincaré exactly 100 years ago, the conjecture holds that the 3-sphere is unique among 3-manifolds; no other 3-manifold shares the properties that make it so simple. The 3-manifolds that are more complicated than the 3-sphere have boundaries that you can run up against like a brick wall, or multiple connections from one region to another, like a path through the woods that splits and later rejoins. The Poincaré conjecture states that the 3-sphere is the only compact 3-manifold that lacks all those complications. Any three-dimensional object that shares those properties with the sphere can therefore be morphed into the same shape as a 3-sphere; so far as topologists are concerned, the object *is* just another copy of the 3-sphere. Perelman’s proof also answers the third of our questions: it completes work that classifies all the types of 3-manifolds that exist.

It takes some mental gymnastics to imagine what a 3-sphere is like—it is not simply a sphere in the everyday sense of the word [see box on pages 98 and 99]. But it has many properties in common with the 2-sphere, which we are all familiar with: If you take a spherical balloon, the rubber of the balloon forms a 2-

## Overview/*Proving Poincaré*

- For 100 years, mathematicians have been trying to prove a conjecture that was first proposed by Henri Poincaré relating to an object known as the three-dimensional sphere, or 3-sphere. The conjecture singles out the 3-sphere as being unique among all three-dimensional objects, or manifolds.
- A proof of the Poincaré conjecture has finally come, with the work of a young Russian mathematician, Grigori Perelman. His analysis also completes a major research program that classifies all possible three-dimensional manifolds.
- Our universe might have the shape of a 3-sphere. The mathematics has other intriguing connections to particle physics and Einstein’s theory of gravity.

sphere. The 2-sphere is two-dimensional because only two coordinates—latitude and longitude—are needed to specify a point on it. Also, if you take a very small disk of the balloon and examine it with a magnifying glass, the disk looks a lot like one cut from a flat two-dimensional plane of rubber. It just has a slight curvature. To a tiny insect crawling on the balloon, it would seem like a flat plane. Yet if the insect traveled far enough in what would seem to it to be a straight line, eventually it would arrive back at its starting point.

Similarly, a gnat in a 3-sphere—or a person in one as big as our universe!—perceives itself to be in “ordinary” three-dimensional space. But if it flies far enough in a straight line in any direction, eventually it will circumnavigate the 3-sphere and find itself back where it started, just like the insect on the balloon or someone taking a trip around the world.

2002, when Perelman, a mathematician at the Steklov Institute of Mathematics at St. Petersburg, posted a paper on the [www.arxiv.org](http://www.arxiv.org) Web server that is widely used by physicists and mathematicians as a clearinghouse of new research. The paper did not mention the Poincaré conjecture by name, but topology experts who looked at it immediately realized the paper’s relevance to that theorem. Perelman followed up with a second paper in March 2003, and from April to May that year he visited the U.S. to give a series of seminars on his results at the Massachusetts Institute of Technology and Stony Brook University. Teams of mathematicians at nearly a dozen leading institutes began poring over his papers, verifying their every detail and looking for errors.

At Stony Brook, Perelman gave two weeks of formal and informal lectures, talking from three to six hours a day. “He

ous a peer review as any paper gets.)

Perelman’s work extends and completes a program of research that Richard S. Hamilton of Columbia University explored in the 1990s. The Clay Institute recognized Hamilton’s work with a research award in late 2003. Perelman’s calculations and analysis blow away several roadblocks that Hamilton ran into and could not overcome.

If, as everyone expects, Perelman’s proof is correct, it actually completes a much larger body of work than the Poincaré conjecture. Launched by William P. Thurston—now at Cornell University—the Thurston geometrization conjecture provides a full classification of all possible 3-manifolds. The 3-sphere, unique in its sublime simplicity, anchors the foundation of this magnificent classification. Had the Poincaré conjecture been false—that is, if there were many spaces as “simple” as

## TEAMS OF MATHEMATICIANS at nearly a dozen leading institutes began poring over Perelman’s papers, verifying every detail and LOOKING FOR ERRORS.

Spheres exist for dimensions other than three as well. The 1-sphere is also familiar to you: it is just a circle (the rim of a disk, not the disk itself). The  $n$ -dimensional sphere is called an  $n$ -sphere.

### Proving Conjectures

AFTER POINCARÉ proposed his conjecture about the 3-sphere, half a century went by before any real progress was made in proving it. In the 1960s mathematicians proved analogues of the conjecture for spheres of five dimensions or more. In each case, the  $n$ -sphere is the unique, simplest manifold of that dimensionality. Paradoxically, this result was easier to prove for higher-dimensional spheres than for those of four or three dimensions. The proof for the particularly difficult case of four dimensions came in 1982. Only the original three-dimensional case involving Poincaré’s 3-sphere remained open.

A major step in closing the three-dimensional problem came in November

answered every question that arose, and he was very clear,” says mathematician Michael Anderson of Stony Brook. “No one has yet raised any serious doubts.” One more comparatively minor step has to be proved to complete the result, Anderson says, “but there are no real doubts about the validity of this final piece.” The first paper contains the fundamental ideas and is pretty well accepted as being verified. The second paper contains applications and more technical arguments; its verification has not reached the level of confidence achieved for the first paper.

The Poincaré conjecture has a \$1-million reward on offer for its proof: it is one of seven such “Millennium Problems” singled out in 2000 by the Clay Mathematics Institute in Cambridge, Mass. Perelman’s proof has to be published and withstand two years of scrutiny before he becomes eligible for the prize. (The institute might well decide that its posting on the Web server qualifies as “published” because the result is undergoing as rigor-

a sphere—the classification of 3-manifolds would have exploded into something infinitely more complicated than that proposed by Thurston. Instead, with Perelman’s and Thurston’s results, we now have a complete catalogue of all the possible shapes that a three-dimensional space can take on—all the shapes allowed by mathematics that our universe (considering just space and not time) could have.

### Rubber Doughnuts

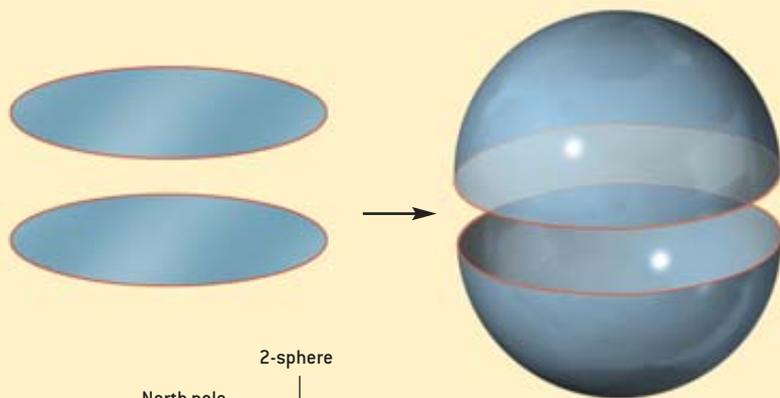
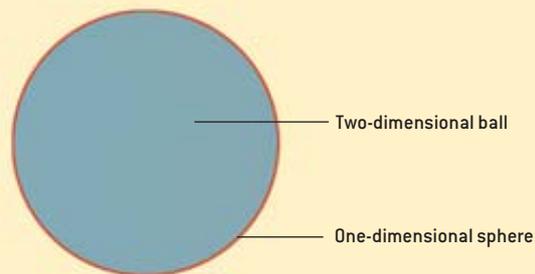
TO UNDERSTAND the Poincaré conjecture and Perelman’s proof in greater depth, you have to know something about topology. In that branch of mathematics the exact shape of an object is irrelevant, as if it were made of play dough that you could stretch, compress and bend to any extent. But why should we care about objects or spaces made of imaginary play dough? The reason relates to the fact that the exact shape of an object—the distance from one point on it to another—is a level of structure, which is

# Multidimensional Music of Spheres

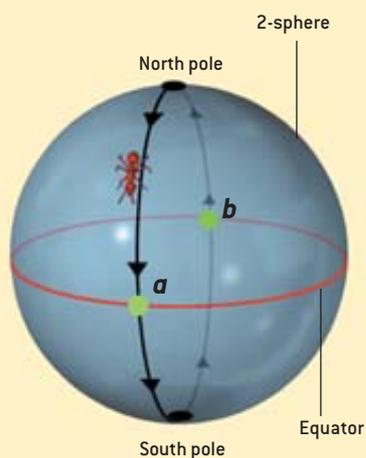
The 3-sphere at the heart of Poincaré’s conjecture takes a bit of effort to visualize. Mathematicians who prove theorems about higher-dimensional spaces do not have to visualize them. They make do with abstract properties, guided by intuitive notions based on

analogies to lower dimensions (but being careful not to take the analogies literally). Others, too, can form an idea of what higher-dimensional objects are like by working up from familiar lower-dimensional examples. The 3-sphere is a case in point.

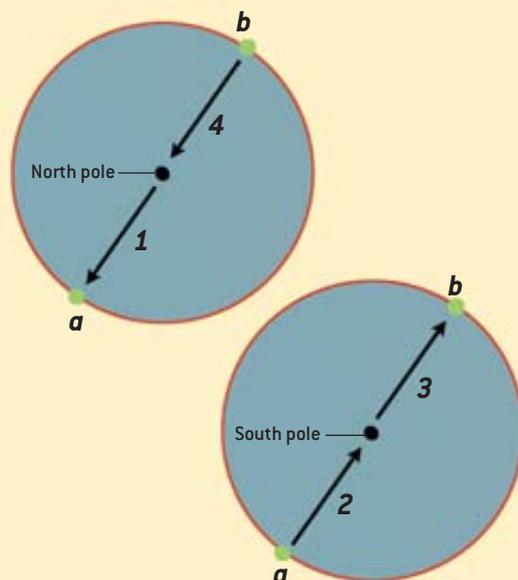
**1** Start by considering a disk with a circle forming its boundary. To mathematicians, the disk is a “two-dimensional ball” and the circle is a “one-dimensional sphere.” Also, a “ball” of whatever dimensionality is the filled-in object, analogous to a baseball, and a “sphere” is the surface of a ball, analogous to a balloon. The circle is one-dimensional because it takes just a single number to specify a location on it.



**2** Now we can build the two-dimensional sphere out of two copies of the disk. Distort one disk into a hemisphere like the Northern Hemisphere and distort the other to be like the Southern Hemisphere. Then glue the two hemispheres together at their boundary, which becomes the equator. Voilà: the 2-sphere.



**3** Imagine an ant starting out from the North Pole, walking along the great circle formed by the International Date Line and the meridian that passes through Greenwich, England (left). If we map that path back onto the two disks (right), we see that the ant travels in a straight line [1] out to the edge of the northern disk (a). Then it crosses to the corresponding point on the southern disk and continues across that disk in a straight line [2 and 3]. When it reaches the edge again (b), it crosses back to the northern disk and continues on to its starting point, the North Pole [4]. We have followed its path as it circumnavigates the 2-sphere by tracking it on the disks. The only tricky part is figuring out that the direction of travel appears to reverse when it crosses from one disk to the other.



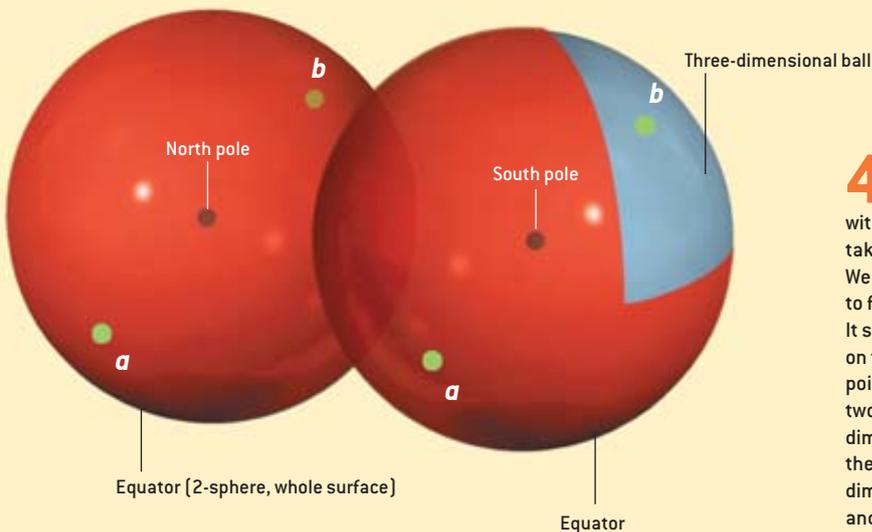
called the geometry of the object. By considering a play-dough object, topologists discover which properties of the object are so fundamental that they exist independently of its geometric structure. Studying topology is like discovering which properties humans have in general by considering the properties of a “play-dough person” who can be morphed into any particular human being.

If you have read any popular account of topology, you have probably encountered the hoary old truism that a cup and

a doughnut are indistinguishable to a topologist. (The saying refers to a ring-shaped doughnut, not the solid, jam-filled kind.) The point is that you can morph the play-dough cup into a doughnut shape simply by smushing the clay around, without having to cut out any holes or glue any patches together [see illustration on page 100]. A ball, on the other hand, can be turned into a doughnut only by either boring a hole through its middle or stretching it into a cylinder and gluing the ends together. Because such cutting or glu-

ing is needed, a ball is not the same as a doughnut to a topologist.

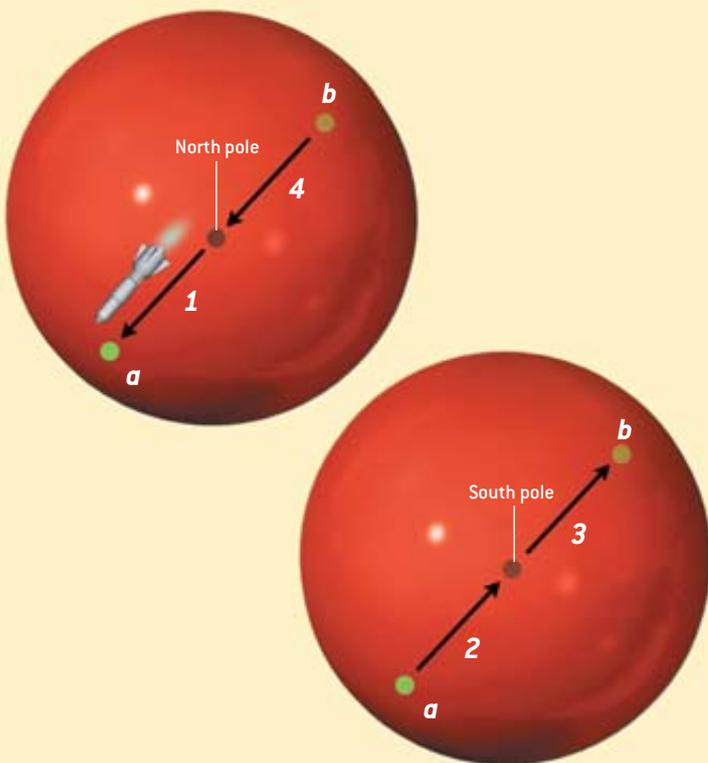
What interests topologists most are the surfaces of the ball and the doughnut, so instead of imagining a solid we should imagine a balloon in both cases. The topologies are still distinct—the spherical balloon cannot be morphed into the ring-shaped balloon, which is called a torus. Topologically, then, a sphere and a torus are distinct entities. Early topologists set out to discover how many other topologically distinct entities exist and how they



**4** Then consider our 2-sphere and the three-dimensional volume it contains (a “three-dimensional ball”) and do with the ball and sphere what we did with the circle and disk: take two copies and glue the boundaries together. We cannot visualize how to distort the balls in four dimensions to form some analogue of hemispheres, but we don’t need to. It suffices to know that corresponding points on the surfaces—on the 2-spheres—are joined together, like corresponding points on the circles were. The result of joining together the two balls is the 3-sphere, which is the “surface” of a four-dimensional ball. (In four dimensions—where the 3-sphere and the 4-ball really live—the “surface” of an object is three-dimensional.) We can call one ball the northern hemisphere and the other the southern. The north pole is at the center of the northern ball (like the north pole was marked at the center of the northern disk).

**5** Next imagine these balls are large, empty regions of space, and someone sets out in a rocket ship from the north pole. Eventually she reaches the “equator” (1), which is the entire sphere surrounding the northern ball. At the equator she crosses over to the southern hemisphere and travels on a straight line through its center (the south pole) to the opposite side of the equator (2 and 3). There she crosses back to the northern hemisphere and travels back to the north pole, her starting point (4). We have just imagined someone traveling along the surface of a four-dimensional ball, circumnavigating it! This 3-sphere, consisting of two balls joined on their spherical surfaces, is the space to which the Poincaré conjecture applies. Our universe might have the shape of a 3-sphere.

The process can be continued to five dimensions—to make a 4-sphere—but it becomes even harder to visualize what is going on. Similarly, any given  $n$ -sphere can be constructed from two  $n$ -balls: just glue together the boundaries of the two balls. Each boundary is an  $(n-1)$ -sphere, just as the boundary of a disk (a 2-ball) is a circle (a 1-sphere). The result is an  $n$ -sphere, which encloses an  $(n+1)$ -ball.



could be characterized. For two-dimensional objects, which are also called surfaces, the answer is neat and tidy: it all boils down to how many “handles” a surface has.

By the end of the 19th century, mathematicians understood how to classify surfaces. Out of all the surfaces, the sphere, they knew, had a unique simplicity. Naturally they started wondering about three-dimensional manifolds. To start with, was the 3-sphere unique in its simplicity, analogous to the 2-sphere?

The century-long history that follows from that elementary question is littered with false steps and false proofs.

Henri Poincaré tackled this question head-on. He was one of the two foremost mathematicians who were active at the turn of the 20th century (the other being David Hilbert). Poincaré has been called the last universalist—he was at ease in all branches of mathematics, both pure and applied. In addition to advancing numerous areas of mathematics, he contributed to the theories of celestial mechanics and

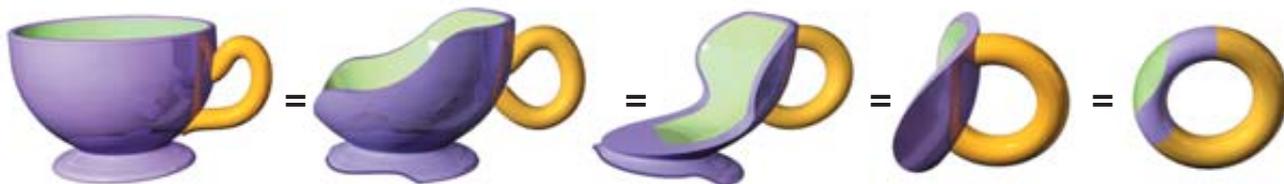
electromagnetism as well as to the philosophy of science (about which he wrote several widely read popular books).

Poincaré largely created the branch of mathematics called algebraic topology. Around 1900, using techniques from that field, he formulated a measure of an object’s topology, called homotopy. To determine a manifold’s homotopy, imagine that you embed a closed loop in the manifold [see box on next page]. The loop can be wound around the manifold in any possible fashion. We then ask, Can the

# TOPOLOGY OF SURFACES

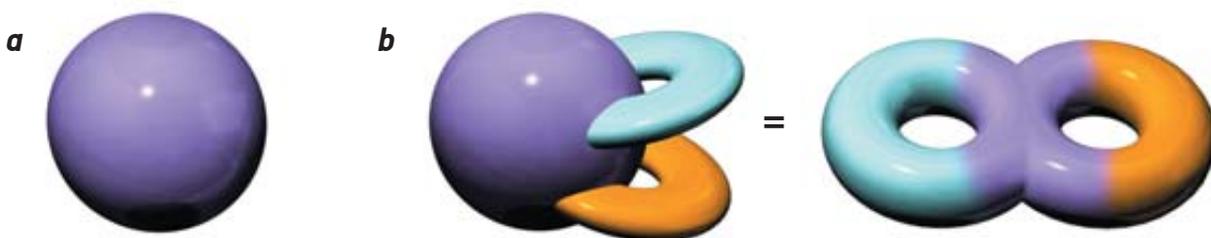
**IN TOPOLOGY** the exact shape, or geometry, of an object is not important. It is as if everything is made of play dough or rubber and can be morphed by stretching, compressing and twisting. Cutting and

joining, however, are forbidden. Thus, in topology any object with a single hole, such as the coffee cup at the far left, is equivalent to the doughnut at the far right.



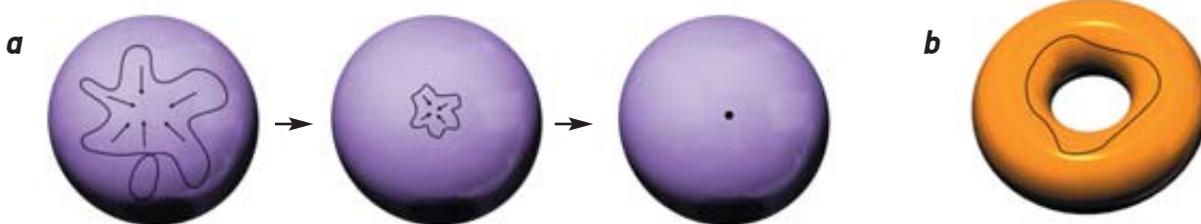
**EVERY POSSIBLE** two-dimensional manifold, or surface (restricting to so-called compact, orientable ones), can be constructed by taking a sphere (akin to a balloon, *a*) and adding handles.

The addition of one handle yields the genus-1 surface, or torus, which is the surface of the doughnut shape (*above right*). Adding two handles yields the genus-2 surface (*b*) and so on.



**2-SPHERE** is unique among surfaces, in that any closed loop embedded on a 2-sphere can be shrunk down to a point (*a*). In contrast, a loop on a torus can get “caught” around the hole in the middle (*b*). Every surface except for the 2-sphere has handles on

which the loop can get caught. The Poincaré conjecture proposes that the 3-sphere is similarly unique among all three-dimensional manifolds: any loop on it can be shrunk to a point, but on every other 3-manifold, the loop can get caught.



loop always be shrunk down to a point, just by moving it around, without ever lifting a piece of it out of the manifold? On a torus the answer is no. If the loop runs around the circumference of the torus, it cannot be shrunk to a point—it gets caught on the inner ring of the doughnut. Homotopy is a measure of all the different ways a loop can get caught.

On an  $n$ -sphere, no matter how convoluted a path the loop takes, it can always be untangled and shrunk to a point. (The loop is allowed to pass through itself during these manipulations.) Poincaré speculated that the only 3-manifold

on which every possible loop can be shrunk to a point was the 3-sphere itself, but he could not prove it. In due course this proposal became known as the Poincaré conjecture. Over the decades, many people have announced proofs of the conjecture, only to be proved wrong. (For clarity, here and throughout I ignore two complications: so-called nonorientable manifolds and manifolds with edges. For example, the Möbius band, a ribbon that is twisted and joined in a loop, is nonorientable. A sphere with a disk cut out from it has an edge. The Möbius band also has an edge.)

## Geometrization

PERELMAN'S PROOF is the first to withstand close scrutiny. His approach to analyzing three-dimensional manifolds is related to a procedure called geometrization. Geometry relates to the actual shape of an object or manifold: for geometry, an object is made not of play dough but of ceramic. A cup, for example, has a different geometry than a doughnut; its surface curves in different ways. It is said that the cup and the doughnut are two examples of a topological torus (provided the cup has one handle) to which different geometries have been assigned.

# GEOMETRIZATION

To gain a sense of how geometrization served to help Perelman, consider how geometry can be used to classify 2-manifolds, or surfaces. Each topological surface is assigned a special, unique geometry: the one for which the curvature of the surface is spread completely evenly about the manifold. For the sphere, that unique geometry is the perfectly spherical sphere. An eggshell shape is another possible geometry for a topological sphere, but it does not have curvature evenly spread throughout: the small end of the egg is more curved than the big end.

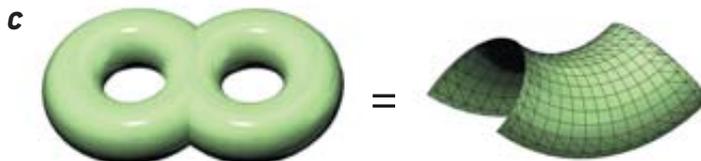
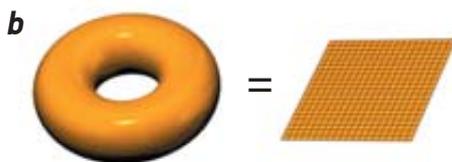
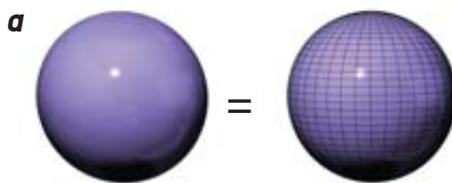
The 2-manifolds form three geometric types [see box at right]. The sphere has what is called positive curvature, the shape of a hilltop. The geometrized torus is flat; it has zero curvature, like a plain. All the other manifolds, with two or more handles, have negative curvature. Negative curvature is like the shape of a mountain pass or a saddle: going from front to back, a saddle curves up; from left to right, it curves down. Poincaré (who else?), along with Paul Koebe and Felix Klein (after whom the Klein bottle is named), contributed to this geometric classification, or geometrization, of 2-manifolds.

It is natural to try to apply similar methods to 3-manifolds. Is it possible to find unique geometries for each topological 3-manifold, for which curvature is spread evenly throughout the manifold?

It turns out that 3-manifolds are far more complicated than 2-manifolds. Most 3-manifolds cannot be assigned a uniform geometry. Instead they have to be cut up into pieces, each piece having a different canonical geometry. Furthermore, instead of three basic geometries, as with 2-manifolds, the 3-manifold pieces can take any of eight canonical geometries. The cutting up of each 3-manifold is somewhat analogous to the factorization of a number into a unique product of prime factors.

This classification scheme was first conjectured by Thurston in the late 1970s. He and his colleagues proved large swaths of the conjecture, but crucial points that the entire system depended on remained beyond their grasp, including the part that embodied the Poincaré conjecture. Was the 3-sphere unique? An answer to that question and completion of the

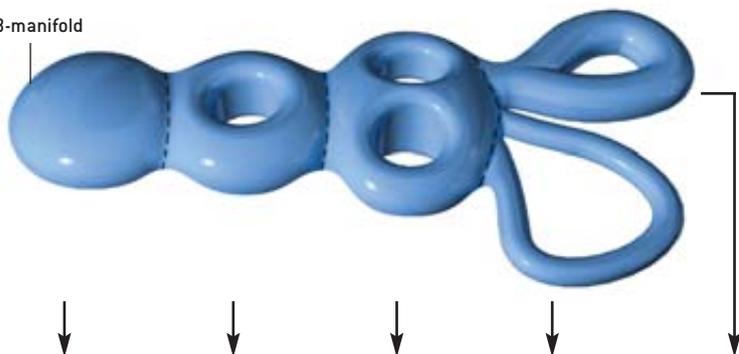
**2-MANIFOLDS** can be classified by “uniformizing” or “geometrizing” them, which means assigning them a specific geometry, or rigid shape. In particular, each can be morphed into a shape that has its curvature evenly distributed. The sphere [a] is the unique shape having constant positive curvature, meaning at every point it is curved like a hilltop. The torus [b] can be made flat—that is, with zero curvature throughout. To see this, imagine cutting the torus and straightening it out to form a cylinder. Then cut along the cylinder and unroll it to form a flat rectangular plane. The torus has thus been mapped to a flat plane. Surfaces of genus-2 and higher [c] can be given constant negative curvature, with other details depending on how many handles are present. Here the constant negative curvature surface is represented by the saddle shape.



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**CLASSIFICATION OF 3-MANIFOLDS**, which is similar to that of 2-manifolds but far more complicated, is completed by Perelman's work. In general, a 3-manifold has to be divided into pieces, each of which can be morphed into one of eight different canonical three-dimensional geometries. The blue-colored example below (drawn in cartoon form as a 2-manifold) consists of equivalents to five of them: constant positive [a], zero [b] and constant negative [c] curvature 3-geometries, as well as the “product” of the 2-sphere and a circle [d] and of the negative curvature surface and a circle [e].

3-manifold



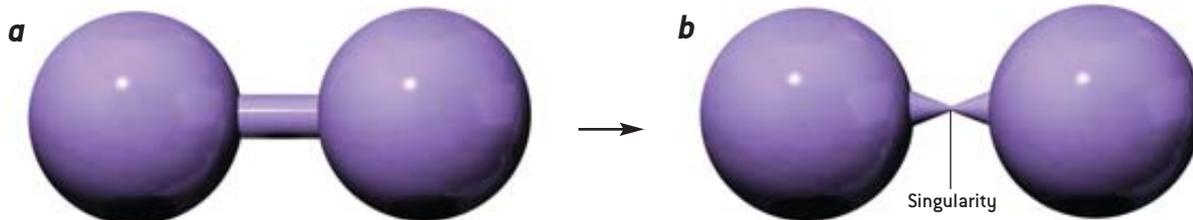
EXAMPLES OF CANONICAL 3-GEOMETRIES

<b>a</b>	<b>b</b>	<b>c</b>	<b>d</b>	<b>e</b>

## DEALING WITH SINGULARITIES

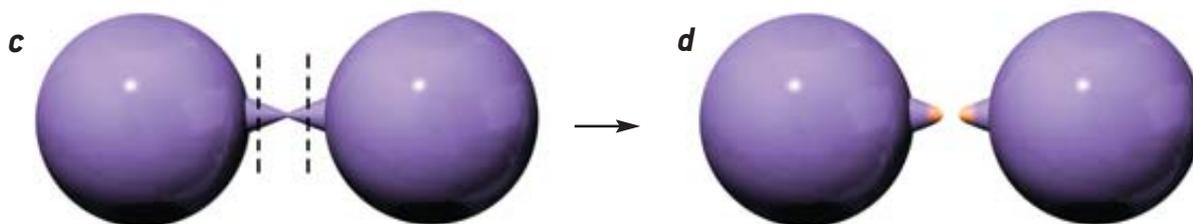
**EFFORTS TO APPLY** an equation called the Ricci flow to prove Poincaré's conjecture and to geometrize 3-manifolds hit a roadblock before Perelman came along. The Ricci flow, which gradually changes the shape of a 3-manifold, occasionally runs into trouble called singularities.

One instance is when part of the manifold has a dumbbell shape—two spheres joined by a tube (a). The tube can become pinched down to a point, spoiling the manifold's properties (b). Another singularity, called the cigar singularity, was also believed to be possible.



**"SURGERY" CAN HANDLE** the singularities that arise in the Ricci flow, as shown by Perelman's work. When a region of the manifold starts to pinch down, a small region on each side of the pinch can be cut out (c). The cuts are then capped by small spheres, and the Ricci flow

is continued (d). This process may have to be repeated several times if other regions subsequently pinch down, but Perelman showed that the process terminates eventually. He also showed that cigar singularities never arise.



Thurston program have come only with Perelman's papers.

How might we try to geometrize a manifold—that is, give it a uniform curvature throughout? One way is to start with some arbitrary geometry, perhaps like an eggshell shape with various lumps and indentations, and then smooth out all the irregularities. Hamilton began such a program of analysis for 3-manifolds in the early 1990s, using an equation called the Ricci flow (named after mathematician Gregorio Ricci-Curbastro), which has some similarities to the equation that governs the flow of heat. In a body with hot and cold spots, heat naturally flows from the warmer regions to the cooler ones, until the temperature is uniform everywhere. The Ricci flow equation has a similar effect on curvature, morphing a manifold to even out all the bumps and hollows. If you began with an egg, it would gradually become perfectly spherical.

Hamilton's analysis ran into a stumbling block: in certain situations the Ricci flow would cause a manifold to pinch down to a point. (This is one way that the Ricci flow differs from heat flow. The places that are pinched are like points that

manage to acquire infinite temperature.) One example was when the manifold had a dumbbell shape, like two spheres connected by a thin neck. The spheres would grow, in effect drawing material from the neck, which would taper to a point in the middle [see box above]. Another possible example arose when a thin rod stuck out from the manifold; the Ricci flow might produce a trouble called a cigar singularity. When a manifold is pinched in this way, it is called singular—it is no longer a true three-dimensional manifold. In a true three-dimensional manifold, a small region around any point looks like a small region of ordinary three-dimensional space, but this property fails at pinched points. A way around this stumbling block had to wait for Perelman.

Perelman came to the U.S. as a postdoctoral student in 1992, spending semesters at New York University and Stony Brook, followed by two years at the University of California at Berkeley. He quickly made a name for himself as a brilliant young star, proving many important and deep results in a particular branch of geometry. He was awarded a prize from the European Mathematical

Society (which he declined) and received a prestigious invitation to address the International Congress of Mathematicians (which he accepted). In spring 1995 he was offered positions at a number of outstanding mathematics departments, but he turned them all down to return to his home in St. Petersburg. "Culturally, he is very Russian," commented one American colleague. "He's very unmaterialistic."

Back in St. Petersburg, he essentially disappeared from mathematicians' radar screens. The only signs of activity, after many years, were rare occasions when he e-mailed former colleagues, for example, to point out errors in papers they had posted on the Internet. E-mails inquiring about his pursuits went unanswered.

Finally, in late 2002 several people received an e-mail from him alerting them to the paper he had posted on the mathematics server—just a characteristically brief note saying they might find it of interest. That understatement heralded the first stage of his attack on the Poincaré conjecture. In the preprint, along with his Steklov Institute affiliation, Perelman acknowledged support in the form of money he had saved from his U.S. postdoctoral positions.

POINCARÉ [seated, talking to Marie Curie] attended the first Solvay Physics Conference in Brussels, Belgium, in October 1911. Behind stand Ernest Rutherford, Heike Kamerlingh Onnes [who discovered superconductivity earlier that year] and Albert Einstein. This conference may have been the only time that Einstein and Poincaré met. Poincaré died nine months later.

In his paper, Perelman added a new term to the Ricci flow equation. The modified equation did not eliminate the troubles with singularities, but it enabled Perelman to carry the analysis much further. With the dumbbell singularities he showed that “surgery” could be performed: Snip the thin tube on each side of the incipient pinch and seal off the open tube on each dumbbell ball with a spherical cap. Then the Ricci flow could be continued with the surgically altered manifold until the next pinch, for which the same procedure could be applied. He also showed that cigar singularities could not occur. In this way, any 3-manifold could be reduced to a collection of pieces, each having a uniform geometry.

When the Ricci flow and the surgery are applied to all possible 3-manifolds, any manifold that is as “simple” as a 3-sphere (technically, that has the same homotopy as a 3-sphere) necessarily ends up with the same uniform geometry as a 3-sphere. That result means that topologically, the manifold in question *is* a 3-sphere. Rephrasing that, the 3-sphere is unique.

Beyond proving Poincaré’s conjecture, Perelman’s research is important for the innovative techniques of analysis it has introduced. Already mathematicians are posting papers that build on his work or apply his techniques to other problems. In addition, the mathematics has curious connections to physics. The Ricci flow used by Hamilton and Perelman is related to something called the renormalization group, which specifies how interactions change in strength depending on the energy of a collision. For example, at low energies the electromagnetic interaction has a strength characterized by the number 0.0073 (about  $\frac{1}{137}$ ). If two electrons collide head-on at nearly the speed of light, however, the strength is closer to 0.0078.

Increasing the collision energy is equivalent to studying the force at a shorter distance scale. The renormalization group is therefore like a microscope with a magnification that can be turned up or



down to examine a process at finer or coarser detail. Similarly, the Ricci flow is like a microscope for looking at a manifold at a chosen magnification. Bumps and hollows visible at one magnification disappear at another. Physicists expect that on a scale of about  $10^{-35}$  meter, or the Planck length, the space in which we live will look very different—like a “foam” with many loops and handles and other topological structures [see “Atoms of Space and Time,” by Lee Smolin; SCIENTIFIC AMERICAN, January]. The mathematics that describes how the physical forces change is very similar to that which describes geometrization of a manifold.

Another connection to physics is that the equations of general relativity, which describe the workings of gravity and the

large-scale structure of the universe, are closely related to the Ricci flow equation. Furthermore, the term that Perelman added to the basic flow used by Hamilton arises in string theory, which is a quantum theory of gravity. It remains to be seen if his techniques will reveal interesting new information about general relativity or string theory. If that is the case, Perelman will have taught us not only about the shapes of abstract 3-spaces but also about the shape of the particular space in which we live. SA

*Graham P. Collins, a staff writer and editor, has degrees in mathematics and physics. For additional information on the Poincaré conjecture, visit [www.sciam.com/ontheweb](http://www.sciam.com/ontheweb)*

#### MORE TO EXPLORE

**The Poincaré Conjecture 99 Years Later: A Progress Report.** John W. Milnor. February 2003. Available at [www.math.sunysb.edu/~jack/PREPRINTS/poiproof.pdf](http://www.math.sunysb.edu/~jack/PREPRINTS/poiproof.pdf)

**Jules Henri Poincaré** [biography]. October 2003. Available at [www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Poincare.html](http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Poincare.html)

**Millennium Problems.** The Clay Mathematics Institute: [www.claymath.org/millennium/](http://www.claymath.org/millennium/)

**Notes and commentary on Perelman’s Ricci flow papers.** Compiled by Bruce Kleiner and John Lott. Available at [www.math.lsa.umich.edu/research/ricciflow/perelman.html](http://www.math.lsa.umich.edu/research/ricciflow/perelman.html)

**Topology.** Eric W. Weisstein in *MathWorld—A Wolfram Web Resource*. Available at [mathworld.wolfram.com/Topology.html](http://mathworld.wolfram.com/Topology.html)