

# On Helly Hypergraphs with Variable Intersection Sizes\*

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## Abstract

A hypergraph  $\mathcal{H}$  is said to be  $p$ -Helly when every  $p$ -wise intersecting partial hypergraph  $\mathcal{H}'$  of  $\mathcal{H}$  has nonempty total intersection. Such hypergraphs have been characterized by Berge and Duchet in 1975, and since then they have appeared in the literature in several contexts, especially for the case  $p = 2$ , in which they are referred simply as Helly hypergraphs. An interesting generalization of  $p$ -Helly hypergraphs due to Voloshin takes into account not only the number of intersecting sets, but also the intersection sizes: we say that a hypergraph  $\mathcal{H}$  is  $(p, q, s)$ -Helly when every  $p$ -wise  $q$ -intersecting partial hypergraph  $\mathcal{H}'$  of  $\mathcal{H}$  has total intersection of cardinality at least  $s$ . In this work we propose a characterization for  $(p, q, s)$ -Helly hypergraphs. This characterization leads to an efficient algorithm to recognize such hypergraphs when  $p$  and  $q$  are fixed parameters.

**Keywords:** Helly Property, Helly Hypergraphs, Intersecting Sets

## 1 Introduction

A *hypergraph*  $\mathcal{H}$  is an ordered pair  $(V(\mathcal{H}), E(\mathcal{H}))$  where  $V(\mathcal{H}) = \{v_1, \dots, v_n\}$  is a finite set of *vertices* and  $E(\mathcal{H}) = \{E_1, \dots, E_m\}$  is a set of nonempty *edges* (or *hyperedges*)  $E_i \subseteq V(\mathcal{H})$  where  $V(\mathcal{H}) = \cup_{1 \leq i \leq m} E_i$ . For a set  $J \subseteq \{1, \dots, m\}$ , the hypergraph  $\mathcal{H}'$  such that  $V(\mathcal{H}') = \cup_{j \in J} E_j$  and  $E(\mathcal{H}') = \{E_j : j \in J\}$  is the *partial hypergraph of  $\mathcal{H}$  generated by the set  $J$* . If every edge of a hypergraph

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contains  $k$  vertices, for an integer  $k > 0$ , then  $\mathcal{H}$  is said to be  $k$ -uniform. A graph is a 2-uniform hypergraph.

A hypergraph  $\mathcal{H}$  is said to be  $p$ -Helly when every  $p$ -wise intersecting partial hypergraph  $\mathcal{H}'$  of  $\mathcal{H}$  has nonempty total intersection. Such hypergraphs have been characterized by Berge and Duchet in 1975 [2], and since then they have appeared in the literature in several contexts, especially for the case  $p = 2$ , in which they are referred simply as *Helly hypergraphs*.

In this work we investigate a generalization of  $p$ -Helly hypergraphs. This generalization was originally defined by Voloshin in [13] in the context of hypergraph coloring, and takes into account not only the number of intersecting sets, but also the intersection sizes. For positive integers  $p, q$ , and  $s$ , a hypergraph  $\mathcal{H}$  is said to be a  $(p, q, s)$ -Helly hypergraph if every  $p$ -wise  $q$ -intersecting partial hypergraph  $\mathcal{H}' \subseteq \mathcal{H}$  has total intersection of cardinality at least  $s$ . In other words,  $q$  and  $s$  are additional parameters such that  $q$  indicates the minimum number of elements required in the intersection of every group with  $p$  or fewer edges of  $\mathcal{H}'$ , and  $s$  the minimum number of elements which must be present in the total intersection of  $\mathcal{H}'$ . In this notation,  $p$ -Helly hypergraphs are precisely  $(p, 1, 1)$ -Helly hypergraphs, and Helly hypergraphs are  $(2, 1, 1)$ -Helly hypergraphs.

In this work we propose a characterization for  $(p, q, s)$ -Helly hypergraphs, which generalizes Berge and Duchet's theorem on  $p$ -Helly hypergraphs. As we shall see, this characterization leads to a polynomial-time algorithm to recognize such hypergraphs when  $p$  and  $q$  are fixed parameters. For the case  $q = s$ , we present another characterization in terms of induced matchings of a special bipartite graph.

The study of the Helly property applied to general hypergraphs has extensively been considered, e.g. [1] to [13]. The application of Voloshin's concept to the family of cliques of a graph leads to the class of  $(p, q, s)$ -clique-Helly graphs. A characterization of  $(p, q, q)$ -clique-Helly graphs is described in [7].

A well-known corollary of Berge and Duchet's theorem on  $p$ -Helly hypergraphs is that they can be recognized in polynomial time, for fixed  $p$  (see [1] and [2], respectively). For  $(p, q, s)$ -Helly hypergraphs, besides the polynomial-time recognition algorithm for fixed  $p, q$  presented in this work, complexity aspects corresponding to situations where  $p$  or  $q$  are not fixed are considered in [6]. However, it remains open the complexity of recognizing  $(2, q, q)$ -Helly hypergraphs; results on such hypergraphs are found in [12, 13].

Another generalization of the Helly property, called  $k$ -bounded  $p$ -Helly property, forces the size of the subfamilies to be limited by a variable  $k$ . Such a generalization was first studied by Roberts and Spencer in [11]. The problem of recognizing whether a hypergraph is  $k$ -bounded  $p$ -Helly was proved to be NP-hard in [5].

In [9], the notion of *hereditary  $p$ -Helly property* is introduced. Polynomial-time algorithms for checking the hereditary 2-Helly property in hypergraphs are given in [4, 14], and for hereditary  $p$ -Helly property in [8].

Other recent works concerning theoretic and algorithmic aspects of Helly hypergraphs are [3, 10].

The remaining of the text is organized as follows. In Section 3 we present a characterization of  $(p, q, s)$ -Helly hypergraphs. In Section 4 we deal with the natural case  $q = s$ , corresponding to the  $(p, q, q)$ -Helly property. In Section 5 we discuss the computational complexity of the algorithms obtained from the characterizations. The last section contains the conclusions.

## 2 Preliminaries

We start this section by giving some definitions and notation. We say that  $S$  is an  $m$ -set when  $|S| = m$ , an  $m^-$ -set when  $|S| \leq m$ , and an  $m^+$ -set when  $|S| \geq m$ . Throughout the remainder of this work, this notation will be applied to any term standing for a set. The *core* of  $\mathcal{H}$  is defined as  $core(\mathcal{H}) = \bigcap_{1 \leq i \leq m} E_i$ .

A hypergraph is  $p$ -wise  $q$ -intersecting, or simply  $(p, q)$ -intersecting, if every  $p$  or fewer edges of it share at least  $q$  vertices.

We now present some characteristics and examples of  $(p, q, s)$ -Helly hypergraphs.

**Example 2.1** Consider the family  $\mathcal{A}$  of maximal complete subgraphs of the graph of Figure 1, i.e.,  $E(\mathcal{A}) = \{\{a, b, c, d\}, \{a, b, d, e\}, \{a, b, c, f\}, \{a, c, d, g\}\}$ . Then  $\mathcal{A}$  is not  $(3, 2, 2)$ -Helly, because every three edges of  $\mathcal{A}$  share two vertices, while there is only one vertex in common to all edges of  $\mathcal{A}$ . But  $\mathcal{A}$  is  $(3, 2, 1)$ -Helly, because  $a$  is a vertex present in all edges of  $\mathcal{A}$ .

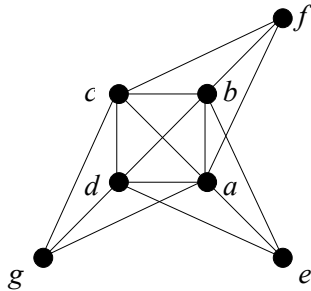


Figure 1: Example of the  $(p, q, s)$ -Helly property.

**Example 2.2** Let  $p \geq 1$  and  $q \geq 0$ . Consider a circular sequence  $a_1, \dots, a_{(p+1)q}$  of distinct points on a circle and a family  $\mathcal{A} = \{\alpha_i : 1 \leq i \leq p+1\}$  of arcs, each  $\alpha_i$  containing  $pq$  circularly consecutive points starting from  $a_{q(i-1)+1}$ . Then  $\mathcal{A}$  is  $(p, q)$ -intersecting but  $core(\mathcal{A}) = \emptyset$ . Thus  $\mathcal{A}$  is not  $(p, q, s)$ -Helly for any  $s \geq 1$ .

The following statements are valid for any hypergraph  $\mathcal{H}$  and positive integers  $p, q, s$ :

- if  $\mathcal{H}$  is  $(p, q, s)$ -Helly then  $\mathcal{H}$  is  $(p + 1, q, s)$ -Helly;
- if  $\mathcal{H}$  is  $(p, q, s)$ -Helly then  $\mathcal{H}$  is  $(p, q + 1, s)$ -Helly;
- if  $\mathcal{H}$  is  $(p, q, s)$ -Helly then  $\mathcal{H}$  is  $(p, q, s - 1)$ -Helly.

### 3 Characterizing $(p, q, s)$ -Helly hypergraphs

We divide the characterization in two cases,  $q \geq s$  and  $s \geq q$ . The following lemma is useful in the sequel.

**Lemma 3.1** *Let  $\mathcal{H}$  be a  $t$ -hypergraph,  $t \geq 2$ . Any non-empty hypergraph with  $t' < t$  hyperedges, each containing at least  $t - 1$  hyperedges of  $\mathcal{H}$ , has core containing at least  $t - t'$  hyperedges of  $\mathcal{H}$ .*

**Proof.** Let  $\mathcal{H}' = \{F_1, \dots, F_{t'}\}$  be a hypergraph where each  $F_i$  contains at least  $t - 1$  hyperedges of a  $t$ -hypergraph  $\mathcal{H}$ , and  $t' < t$ . Since each  $F_i$  does not contain at most one hyperedge of  $\mathcal{H}$ , then at least  $t - t'$  hyperedges of  $\mathcal{H}$  appear in the core of  $\mathcal{H}'$ . ■

Now we can describe the characterization of  $(p, q, s)$ -Helly hypergraphs. First we deal the case  $q \geq s$ .

**Theorem 3.2** *Let  $p, s \geq 1$ ,  $q \geq s$  be integers. A hypergraph  $\mathcal{H}$  is  $(p, q, s)$ -Helly if and only if:*

- (i) *for every family  $\mathcal{S}$  formed by  $p + q - s + 1$  distinct  $s$ -subsets of  $V(\mathcal{H})$ , the partial hypergraph  $\mathcal{H}'$  of  $\mathcal{H}$  formed by all the edges of  $\mathcal{H}$  containing each at least  $p + q - s$  members of  $\mathcal{S}$  satisfies the following statement:  
 $\mathcal{H}'$  is  $(p, q)$ -intersecting  $\Rightarrow |\text{core}(\mathcal{H}')| \geq s$ ;*
- (ii) *every  $(p, q)$ -intersecting partial hypergraph of  $\mathcal{H}$  with  $p + q - s$  or fewer edges has an  $s$ -core.*

**Proof.** The theorem says that, for checking the  $(p, q, s)$ -Helly property in a hypergraph  $\mathcal{H}$ , it is not necessary to check whether every  $(p, q)$ -intersecting partial hypergraph of  $\mathcal{H}$  has  $s$ -core, but it is sufficient to check only few particular partial hypergraphs of  $\mathcal{H}$ . Hence we only need to prove the sufficiency.

Therefore, assume that  $\mathcal{H}$  is not  $(p, q, s)$ -Helly. Then there exists a  $(p, q)$ -intersecting partial hypergraph  $\mathcal{H}'$  of  $\mathcal{H}$  such that  $|\text{core}(\mathcal{H}')| < s$ . If  $|\mathcal{H}'| \leq p + q - s$  then  $\mathcal{H}'$  is a  $(p, q)$ -intersecting partial  $(p + q - s)^-$ -hypergraph of  $\mathcal{H}$  that violates Condition (ii).

Otherwise, write  $\mathcal{H}' = \{E_1, \dots, E_{m'}\}$ , then  $m' \geq p + q - s + 1$ . Assume that  $\mathcal{H}'$  is minimal, that is,  $\mathcal{H}' \setminus \{E\}$  has an  $s^+$ -core, for any  $E \in \mathcal{H}'$ . (If  $\mathcal{H}'$  is not

minimal, one can successively remove hyperedges from  $\mathcal{H}'$  until obtaining either a minimal  $(p + q - s + 1)^+$ -hypergraph or a  $(p + q - s)^-$ -hypergraph violating Condition (ii). For each  $i$ ,  $1 \leq i \leq m'$ , let  $S_i \subseteq \text{core}(\mathcal{H}' \setminus \{E_i\})$  be a  $s$ -subset of vertices such that  $S_i \not\subseteq E_i$  and  $S_i \subseteq E_j$  for every  $j \neq i$ . This means that there exists  $v_i \in S_i$  such that  $v_i \notin E_i$  but  $v_i \in E_j$  for every  $j \neq i$ .

Let  $\mathcal{S} = \{S_1, \dots, S_{p+q-s+1}\}$ . Note that  $\mathcal{S}$  is a family formed by  $p+q-s+1$  distinct  $s$ -subsets of  $V(\mathcal{H})$ . Define  $\mathcal{H}''$  as the hypergraph formed by the edges of  $\mathcal{H}$  each containing  $p+q-s$  edges of  $\mathcal{S}$ . Since  $\mathcal{H}'$  is a partial hypergraph of  $\mathcal{H}''$ ,  $\mathcal{H}''$  does not have an  $s$ -core. Let us show that  $\mathcal{H}''$  is  $(p, q)$ -intersecting.

Consider any partial  $p$ -hypergraph  $\mathcal{H}'''$  of  $\mathcal{H}''$ . By Lemma 3.1,  $\text{core}(\mathcal{H}''')$  contains at least  $q - s + 1$  hyperedges of  $\mathcal{S}$ , say  $S_1, \dots, S_{q-s+1}$ . Note that  $S_1 \cup \{v_i : 2 \leq i \leq q - s + 1\}$  contains exactly  $s + q - s = q$  vertices. This means that  $|\text{core}(\mathcal{H}''')| \geq q$ . Therefore,  $\mathcal{H}''$  is  $(p, q)$ -intersecting and does not have an  $s^+$ -core. This violates Condition (i). ■

Now we deal the case  $s \geq q$ .

**Theorem 3.3** *Let  $p, q \geq 1$ ,  $s \geq q$  be integers. A hypergraph  $\mathcal{H}$  is  $(p, q, s)$ -Helly if and only if:*

- (i) *for every family  $\mathcal{S}$  formed by  $p+1$  distinct  $q$ -subsets of  $V(\mathcal{H})$ , the partial hypergraph  $\mathcal{H}'$  of  $\mathcal{H}$ , formed by all the edges of  $\mathcal{H}$  containing each at least  $p$  members of  $\mathcal{S}$ , has  $s$ -core;*
- (ii) *every  $(p, q)$ -intersecting partial hypergraph of  $\mathcal{H}$  with  $p$  or fewer edges has an  $s$ -core.*

**Proof.** For the necessity we only need to prove that every partial hypergraph  $\mathcal{H}'$  of the Condition (i) is  $(p, q)$ -intersecting. But this is direct from Lemma 3.1.

For the sufficiency, the proof is essentially the same of Theorem 3.2. ■

## 4 The case $q = s$

An interesting case occurs when  $q = s$ . In this situation, Condition (ii) of Theorem 3.2 and 3.3 is trivially satisfied. Hence:

**Corollary 4.1** *Let  $p, q \geq 1$ . A hypergraph  $\mathcal{H}$  is  $(p, q, q)$ -Helly if and only if for every family  $\mathcal{S}$  formed by  $p+1$  distinct  $q$ -subsets of  $V(\mathcal{H})$ , the partial hypergraph of  $\mathcal{H}$ , formed by the edges containing each at least  $p$  edges of  $\mathcal{S}$ , has a  $q$ -core.*

The characterization for  $p$ -Helly hypergraphs [2] corresponds to the case  $q = 1$  of the above corollary.

Another characterization, for  $(p, q, q)$ -Helly hypergraphs, in terms of induced matchings of a bipartite graph is possible. The following definitions will be used.

Let  $G$  be a bipartite graph with color classes  $V_1, V_2$ . A matching  $M \subseteq E(G)$  is said to be *induced* if no two edges of  $M$  are joined by an edge of  $E(G) \setminus M$ .

In addition, we say that  $M$  dominates  $V_2$  if every  $w \in V_2$  is adjacent to some  $M$ -saturated vertex  $u \in V_1$ .

Given a hypergraph  $\mathcal{H} = \{E_1, \dots, E_m\}$  and an integer  $q \geq 1$ , define  $B(\mathcal{H}, q)$  as the bipartite graph with color classes  $V_1, V_2$ , where there is a vertex  $v_i$  in  $V_1$  for every hyperedge  $E_i \in \mathcal{H}$  and a vertex  $u_j$  in  $V_2$  for every  $q$ -subset  $Q_j \subseteq V(\mathcal{H})$ . Finally, there exists an edge between  $v_i \in V_1$  and  $u_j \in V_2$  if and only if  $Q_j \not\subseteq E_i$ .

**Corollary 4.2** *Let  $p, q \geq 1$ . A hypergraph  $\mathcal{H}$  is  $(p, q, q)$ -Helly if and only if there is no induced  $(p+1)^+$ -matching dominating  $V_2$  in  $B(\mathcal{H}, q)$ .*

**Proof.** Let  $\mathcal{H}' = \{F_1, \dots, F_{m'}\}$ ,  $m' \geq p+1$ , be a minimal  $(p, q)$ -intersecting partial hypergraph without  $q$ -core of  $\mathcal{H}$ . For each  $F_i$ , choose one  $q$ -set  $Q_i$  contained in the core of  $\mathcal{H}' \setminus \{F_i\}$ . Observe that the vertices of  $B(\mathcal{H}, q)$ , corresponding to  $F_i$  and  $Q_i$ , for  $1 \leq i \leq m'$ , form an induced  $(p+1)^+$ -matching dominating  $V_2$ . The converse is analogue. ■

## 5 Complexity aspects

Now we analyze the computational complexity of the algorithms obtained by the characterization of  $(p, q, s)$ -Helly hypergraphs we have presented. They terminate within polynomial time whenever  $p$  and  $q$  are fixed.

Theorem 3.2 leads to an algorithm for recognizing  $(p, q, s)$ -Helly hypergraphs, when  $q \geq s$ . The complexity of testing Condition (i) is  $O(pm^p n^{s(p+q-s+1)+1})$ , because there are  $O(n^{s(p+q-s+1)})$  families each with  $p+q-s+1$  distinct  $s$ -subsets of  $V(\mathcal{H})$ . And, for each one, we spend  $(m(n + (p+q-s)s))$  steps to construct every  $\mathcal{H}''$ ,  $O(pnm^p)$  to check if  $\mathcal{H}''$  is  $(p, q)$ -intersecting, and  $O(nm)$  to compute its core. For Condition (ii), the complexity is  $O(pnm^{p+q-s}(p+q-s)^{p+1})$ . The overall time complexity is the sum of both.

Theorem 3.3 deals the case  $s \geq q$ . Analogue to the previous paragraph, the complexity of testing Condition (i) is  $O(mn^{q(p+1)}(n + q(p+1)))$ , because it is not necessary to check if  $\mathcal{H}''$  is  $(p, q)$ -intersecting, and  $O(pnm^p p^{p+1})$  for Condition (ii). For recognizing  $(p, q, q)$ -Helly hypergraphs we use the algorithm which follows directly from Corollary 4.1, and it has the same complexity of Condition (i) of Theorem 3.3.

## 6 Conclusions

We have described a characterization for  $(p, q, s)$ -Helly hypergraphs, which generalizes the characterization of [2] for  $p$ -Helly hypergraphs. The characterization leads to a polynomial-time algorithm for deciding if a hypergraph is  $(p, q, s)$ -Helly, whenever  $p$  and  $q$  are fixed, even if  $s$  is variable. In contrast, recognizing if a hypergraph is  $(p, q, s)$ -Helly hypergraph has been shown to be NP-hard, whenever  $p$  or  $q$  are variable, even if  $s$  is fixed but arbitrary [6].

However, the above results leave unsolved the problem posed by Tuza in [12] to describe a structural characterization for  $(2, q, q)$ -Helly hypergraphs. In particular, what is the complexity of recognizing  $(2, q, q)$ -Helly hypergraphs.

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