Clique-inverse graphs of bipartite graphs[1]

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Abstract

The *clique graph* $K(G)$ of a given graph G is the intersection graph of the collection of maximal cliques of G. Given a family $\mathcal{F}$ of graphs, the *clique-inverse graphs* of $\mathcal{F}$ are the graphs whose clique graphs belong to $\mathcal{F}$. In this work, we describe characterizations for clique-inverse graphs of bipartite graphs, chordal bipartite graphs, and trees. The characterizations lead to polynomial time algorithms for the corresponding recognition problems.

Keywords: intersection graphs, clique graphs, clique-inverse graphs

1 Introduction

Let G be a finite undirected graph with no loops nor multiple edges. Denote the vertex set of G by $V(G)$, and the edge set by $E(G)$. A subgraph H of G is a graph where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a set X of vertices of G, denote by $G[X]$ the *subgraph of G*

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induced by X, that is, the vertex set of $G[X]$ is X and two vertices are adjacent in it if they are so in G.

A *clique* is a subset of vertices inducing a complete subgraph of G, while a *maximal clique* is one not properly contained in any other. The *clique number* $\omega(G)$ of G is the largest order of a clique in G.

A *chord* c is an edge linking two non-consecutive vertices in a cycle. Denote by C_k a cycle with k vertices. A graph is *chordal* if it contains no induced subgraph isomorphic to C_k for $k \geq 4$.

A graph is *bipartite* if its vertex set can be partitioned into two sets U and W such that every edge in $E(G)$ links a vertex of U to a vertex of W. A graph is *chordal bipartite* if it is bipartite and contains no induced subgraph isomorphic to C_{2k} for $k \geq 3$.

The *clique graph* $K(G)$ of G is the intersection graph of the collection of maximal cliques of G. If $H = K(G)$, we say that G is a *clique-inverse graph* of H. Given a family $\mathcal{F}$ of graphs, the family of clique-inverse graphs of $\mathcal{F}$ is defined as

$$K^{-1}(\mathcal{F}) = \{G | K(G) \in \mathcal{F}\}.$$

In [6], Hedetniemi and Slater presented characterizations for clique graphs of triangle-free graphs, bipartite graphs, and trees:

Theorem 1 [6] *Let G be a graph. Then $G \in K(\mathcal{F})$ if and only if $K(G) \in \mathcal{F}$ and any two distinct maximal cliques of G have at most one vertex in common, where $\mathcal{F}$ is one of the following families: triangle-free graphs, bipartite graphs, or trees.* $\square$

Let $\mathcal{I}_k$ be the family of graphs with the following property: any two distinct maximal cliques of a graph in $\mathcal{I}_k$ have at most k vertices in common. Then Hedetniemi and Slater's result can be rewritten as

$$K(\mathcal{F}) = K^{-1}(\mathcal{F}) \cap \mathcal{I}_1,$$

where $\mathcal{F}$ is one of the families cited in the above theorem. Although the problem of characterizing clique graphs of certain families has been studied for several cases, e.g. [1, 2, 5, 6, 7, 11, 15], much less is known about the corresponding inverse problem, which can be stated as follows: given a family $\mathcal{F}$ of graphs, characterize $K^{-1}(\mathcal{F})$, called the family of *clique-inverse graphs* of $\mathcal{F}$. In this work, characterizations are described for clique-inverse graphs of bipartite graphs, chordal bipartite graphs, and trees. The characterizations lead to polynomial time algorithms for solving the corresponding recognition problems.

Clique-inverse graphs were the subjects of [9] and [12]. They are also called *roots* (relative to the clique operator), see e.g. [10]. Clique-inverse graphs of complete graphs are called *clique-complete*. A characterization of the minimal clique-complete graphs with no universal vertex (a vertex adjacent to all other vertices of the graph) has been formulated in [9]. It corresponds to a description of the minimal clique-complete graphs whose maximal cliques do not satisfy the Helly property. In [13], characterizations for clique-inverse graphs of triangle-free graphs and K_4-free graphs are presented in terms of forbidden subgraphs.

The following result is a characterization for clique-inverse graphs of triangle-free graphs. It will be used later:

Theorem 2 [13] *G is a clique-inverse graph of a triangle-free graph if and only if G does not contain as an induced subgraph any of the following graphs: $K_{1,3}$, 4-fan, 4-wheel* (see Figure 1). □

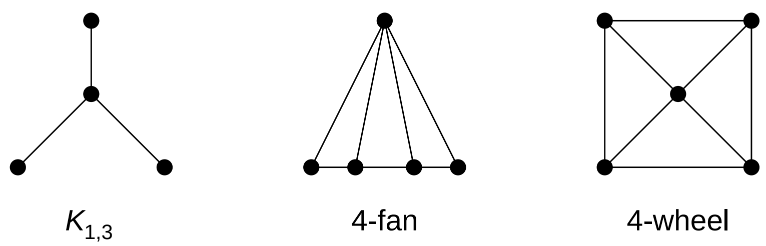

Figure 1: Forbidden subgraphs for clique-inverse graphs of triangle-free graphs.

2 The characterizations

In this section we give complete characterizations for the situations in which $K(G)$ is bipartite, chordal bipartite, or a tree. We begin by analyzing the case in which $K(G)$ is bipartite. The characterization will be formulated in terms of a list of forbidden subgraphs.

Any bipartite graph is triangle-free. Thus, $K^{-1}(BIPARTITE)$ is contained in $K^{-1}(TRIANGLE - FREE)$.

Theorem 3 *A graph G is a clique-inverse graph of a bipartite graph if and only if G does not contain as an induced subgraph any of the following: $K_{1,3}$, 4-fan, 4-wheel, and C_{2k+5} (for all $k \geq 0$).*

Proof. ($\Rightarrow$): Assume by contradiction that G is a clique-inverse graph of a bipartite graph and G contains $S = C_{2k+5}$ as an induced subgraph. Write $S = u_0u_1 \ldots u_pu_0$, where $p = 2k+4$, $k \geq 0$. Clearly, there exists a collection $\mathcal{M} = \{M_0, M_1, \ldots, M_p\}$ of maximal cliques of G such that each edge $e_i = \{u_i, u_{i+1}\}$ of S lies in exactly one of the cliques in $\mathcal{M}$, say e_i lies in M_i (indices are taken circularly in the range $0 \ldots p$). Note that $u_{i+1} \in M_i \cap M_{i+1}$, that is, M_i and M_{i+1} intersect. Thus, $M_0M_1 \ldots M_pM_0$ is an odd cycle in $K(G)$. This is a contradiction, since $K(G)$ is bipartite. On the other hand, by Theorem 2, if G contains either a 4-wheel, a 4-fan, or $K_{1,3}$ as an induced subgraph, then $K(G)$ contains a triangle, another contradiction.
($\Leftarrow$): Assume by contradiction that G does not contain any of the graphs listed in the statement of the theorem as an induced subgraph, and that G is not a clique-inverse graph of a bipartite graph. Then, there exists a chordless odd cycle $C = M_0M_1 \ldots M_{2p}M_0$ in $K(G)$, where $p \geq 1$ and each M_i is a distinct maximal clique of G. Choose C for which p is minimum. There are two possible cases:

Case 1: $p = 1$. Then, $K(G)$ contains a triangle. This implies, by Theorem 2, that G contains either a 4-wheel, a 4-fan, or $K_{1,3}$ as an induced subgraph, a contradiction.

Case 2: $p > 1$. This situation is depicted in Figure 2 (for $p = 2$). Let $u_i \in M_i \cap M_{i+1}$, where indices are taken circularly in the range $0 \ldots 2p$. Note that each u_i belongs to no maximal cliques of G other than M_i and M_{i+1}. Otherwise, if u_i also belongs to a maximal clique M distinct from M_i and M_{i+1}, then $K(G)$ contains a triangle, a contradiction - since the cycle $C = M_0M_1 \ldots M_{2p}M_0$ in $K(G)$ has been taken for $p > 1$ minimum. Thus, the cycle $C_G = u_0u_2 \ldots u_{2p}u_0$ in G is chordless, since the existence of a chord linking non-consecutive vertices u_k and u_j in C_G would imply the existence of a new maximal clique M containing u_k and u_j, distinct from the cliques in the multiset $\{M_k, M_{k+1}, M_j, M_{j+1}\}$. Thus, C_G is a chordless odd cycle with $2p+1 \geq 5$ vertices, which contradicts the assumption that G does not contain C_{2k+5}, $k \geq 0$, as an induced subgraph. $\square$

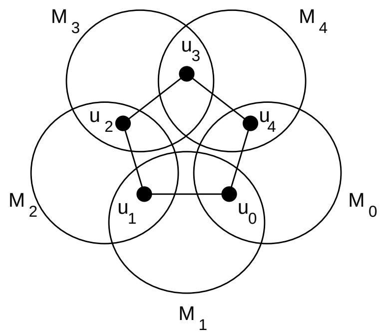

Figure 2: Case 2 of Theorem 3, for $p = 2$.

In order to characterize clique-inverse graphs of chordal bipartite graphs, we employ an additional definition. Let $C = v_0 v_1 \ldots v_k v_0$ ($k \geq 3$) be a cycle in a graph G. We say that C admits an *even division* if there exists a vertex $w \in G \backslash C$ which is adjacent to four distinct vertices $v_i, v_{i+1}, v_j, v_{j+1}$ of C such that $j - (i+1)$ is even, that is, the path $v_{i+1} v_{i+2} \ldots v_j$ has an even number of edges. The indices are taken circularly in the range $0 \ldots k$. See Figure 3.

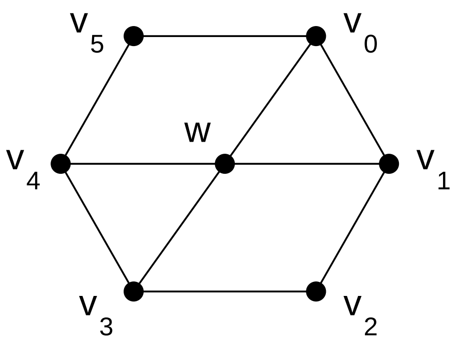

Figure 3: The cycle $v_1 v_2 v_3 v_4 v_5 v_6 v_1$ admits an even division.

The next theorem characterizes $K^{-1}(\mathit{CHORDAL\ BIPARTITE})$ in terms of $K^{-1}(\mathit{BIPARTITE})$:

Theorem 4 *A graph G is a clique-inverse graph of a chordal bipartite graph if and only if $G \in K^{-1}(\mathit{BIPARTITE})$ and every chordless even cycle of G with at least six vertices admits an even division.*

Proof. ($\Rightarrow$): Assume that G is a clique-inverse graph of a chordal bipartite graph, that is, $K(G)$ is chordal bipartite. Clearly, $G \in K^{-1}(\mathit{BIPARTITE})$. Now, let $C = v_0 v_1 \dots v_{2k-1} v_0$ be a chordless even cycle of G with $k \geq 3$. Let M_i be a maximal clique of G containing the edge $\{v_i, v_{i+1}\}$, where indices are taken circularly in the range $1 \dots 2k-1$. It is clear that $M_0 M_1 \dots M_{2k-1} M_0$ is a cycle in $K(G)$, since $v_i \in M_{i-1} \cap M_i$. Assume by contradiction that C does not admit an even division. Then $M_i \cap M_j = \emptyset$ for non-consecutive indices i and j such that $j-(i+1)$ is even. Observe that $M_i \cap M_j = \emptyset$ also holds for non-consecutive i and j such that $j-(i+1)$ is odd, since otherwise $K(G)$ would contain an odd cycle, contradicting $K(G)$ to be bipartite. It follows that $M_0 M_1 \dots M_{2k-1} M_0$ is chordless, $k \geq 3$. This contradicts the fact that $K(G)$ is chordal bipartite.
($\Leftarrow$): Assume that $G \in K^{-1}(\mathit{BIPARTITE})$ and every chordless even cycle of G with at least six vertices admits an even division. Then, $K(G)$ is bipartite. Now, let us show that $K(G)$ does not contain an induced subgraph isomorphic to C_{2k}, for $k \geq 3$. Assume by contradiction that $C = M_0 M_1 \dots M_{2k-1} M_0$ is a chordless cycle in $K(G)$ for $k \geq 3$. Then, there exists a cycle $C_G = v_0 v_1 \dots v_{2k-1} v_0$ in G such that the edge $\{v_i, v_{i+1}\}$ lies in the maximal clique M_i, $0 \leq i \leq 2k-1$, where i is taken circularly in the range $0 \dots 2k-1$. By the assumption, C_G admits an even division. Therefore, let $w \in G \backslash C$ adjacent to four distinct vertices $v_i, v_{i+1}, v_j, v_{j+1}$ of C such that $j-(i+1)$ is even. Observe that v_i belongs to no maximal cliques other than M_{i-1} and M_i, for otherwise $K(G)$ would contain a triangle. Since w, v_i, and v_{i+1} belong to a same maximal clique, it follows that w belongs to at least one of the cliques M_{i-1} and M_i. Analogously, w belongs to at least one of the cliques M_{j-1} and M_j. Thus, some clique of the set $\{M_{i-1}, M_i\}$ intersects at least one clique of the set $\{M_{j-1}, M_j\}$. Since $j-(i+1) > 0$, it follows that there exist two intersecting maximal cliques with non-consecutive indices in the cycle

$C = M_0 M_1 \ldots M_{2k-1} M_0$. This is a contradiction, since C has been assumed to be chordless. □

To conclude this section, let us examine the family $K^{-1}(TREE)$. The following definition will be employed: a *domino* is a graph where every vertex belongs to at most two distinct maximal cliques [8].

Theorem 5 *A graph G is a clique-inverse graph of a tree if and only if G is a chordal domino.*

Proof. ($\Rightarrow$): Assume by contradiction that G is a clique-inverse graph of a tree and G is not chordal, and let $v_0 v_1 \ldots v_k v_0$ be a chordless cycle in G with $k \geq 3$. Then there exist $k+1$ distinct maximal cliques $M_0, M_1, \ldots, M_k$ in G such that the edge $\{v_i, v_{i+1}\}$ lies in M_i and M_i intersects M_{i+1}, where the indices are taken circularly in the range $0 \ldots k$. This implies that $M_0 M_1 \ldots M_k M_0$ is a cycle in $K(G)$. But this is a contradiction, since $K(G)$ is a assumed to be a tree.

Assume now that G is a clique-inverse graph of a tree and G contains a vertex v belonging to more than two maximal cliques of G. Then it follows that the maximal cliques of G containing v induce a clique of size strictly greater than two in $K(G)$. This is another contradiction, since $K(G)$ is assumed to be a tree.

($\Leftarrow$): Assume by contradiction that G is a chordal domino and $K(G)$ is not a tree. Let $M_0 M_1 \ldots M_k M_0$, $k \geq 2$, be a cycle in $K(G)$. There are two possible cases.

Case 1: $k = 2$.

Let $R = M_0 \cap M_1 \cap M_2$. It is clear that $R = \emptyset$, since every vertex of G belongs to at most two maximal cliques. Let $v_{01} \in M_0 \cap M_1$, $v_{02} \in M_0 \cap M_2$, and $v_{12} \in M_1 \cap M_2$. Observe that v_{01}, v_{02}, and v_{12} induce a triangle in G. Therefore, there exists a maximal clique M in G containing v_{01}, v_{02}, and v_{12}. Clearly, $M \neq M_0$, since $v_{12} \in M$ and $v_{12} \notin M_0$. Analogously, $M \neq M_1$. This implies that v_{01} belongs to M_0, M_1, and M, contradicting the fact that every vertex of G belongs to at most two maximal cliques.

Case 2: $k > 2$.

Let $v_i \in M_i \cap M_{i+1}$, where $0 \leq i \leq k$ and indices taken circularly in the range $0 \ldots k$. Then, $C = v_0 v_1 \ldots v_k v_0$ is a cycle in G. The v_i's are distinct, for otherwise, if $v_i = v_j$ for $i \neq j$, then v_i would belong to all the cliques in the multiset $\{M_i, M_{i+1}, M_j, M_{j+1}\}$, which contains at least three distinct elements. Since G is chordal, C has a chord joining v_r and $v_{(r+2)\ mod\ k}$, for some r in the range $0 \ldots k$. Thus, v_r, v_{r+1}, and v_{r+2} induce a triangle in G. This implies that there exists a maximal clique M in G containing these three vertices. Clearly, $M \neq M_r$, since $v_{r+1} \in M$ and $v_{r+1} \notin M_r$. Analogously, $M \neq M_{r+1}$. Thus, v_r belongs to M_r, M_{r+1}, and M. This contradicts the fact that every vertex of G belongs to at most two distinct maximal cliques. □

Corollary 6 *Let G be a graph. Then, G is a clique-inverse graph of a tree if and only if G does not contain as an induced subgraph any of the following graphs: $K_{1,3}$, 4-fan, 4-wheel, C_k (for all $k \geq 4$).*

Proof. If $G \in K^{-1}(TREE)$, then G is a clique-inverse graph of a triangle-free graph. Therefore, by Theorem 2, G does not contain as an induced subgraph any of the following graphs: $K_{1,3}$, 4-fan, 4-wheel. Moreover, by Theorem 5, G is chordal, and thus the first part follows. Conversely, if G does not contain $K_{1,3}$, 4-fan, 4-wheel, or C_k ($k \geq 4$) as an induced subgraph, then G is chordal. Moreover, by Theorem 2, $K(G)$ contains no triangle, which implies that each vertex of G can belong to at most two distinct maximal cliques, that is, G is a domino. Thus, by Theorem 5, G is a clique-inverse graph of a tree. □

3 Algorithms

We start this section by observing that if $K(G)$ has bounded clique number, then $|V(K(G))|$ is $O(n)$, that is, the number of maximal cliques of G is linearly bounded.

Lemma 7 [14] *Let G be a connected graph. If $\omega(K(G)) \leq r$ for a positive constant r, then $|V(K(G))| \leq rn$.*

Proof. Observe that any vertex v of G may belong to at most r maximal cliques, since otherwise the cliques of G containing v would correspond to a clique of size at least $r+1$ in $K(G)$, a contradiction.

Therefore, the number of maximal cliques of G is at most rn, that is, $|V(K(G))| \leq rn$. $\square$

A consequence of the above lemma is the fact that clique-inverse graphs of bipartite graphs have few maximal cliques.

Corollary 8 *Let G be a connected graph. If $K(G)$ is bipartite then G contains at most $2n$ maximal cliques.* $\square$

By using the above observations, we describe below polynomial-time recognition algorithms for the families focused in this work.

Let G be a graph. In order to decide whether or not $K(G)$ is bipartite, first check whether G contains at most $2n$ maximal cliques by applying the algorithm in [16] to $\overline{G}$, which generates all the maximal cliques of G with delay $O(nm)$, where $m = |E(G)|$. This task takes $O(n^2m)$ time. If G has more than $2n$ maximal cliques, then the answer to the question 'Is G in $K^{-1}(BIPARTITE)$?' is clearly 'no'. Otherwise, construct $K(G)$ by taking the maximal cliques generated by the algorithm. This task takes $O(nm)$ time, since G has at most $2n$ maximal cliques, and each intersection test between two cliques takes $O(m)$ time. Finally, verify whether $K(G)$ is bipartite in $O(m)$ time (recall that $|V(K(G))|$ is $O(n)$). Therefore, the entire procedure answers the question 'Is G in $K^{-1}(BIPARTITE)$?' in polynomial time.

In order to decide whether $K(G)$ is chordal bipartite, apply a similar algorithm. The graph $K(G)$ can be constructed in polynomial time. In addition, $K(G)$ can be recognized as a chordal bipartite graph also in polynomial time [4].

Finally, one may use the characterization of Theorem 5 to verify whether $K(G)$ is a tree. Checking chordality and recognizing whether G is a domino can be done in polynomial time.

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