

UNIVERSIDADE FEDERAL FLUMINENSE

NEY PARANAGUÁ DE CARVALHO

**Relative size and *csw*-efficiency:
two new concepts based on *DEA* to guide
management of capital**

NITERÓI

2016

UNIVERSIDADE FEDERAL FLUMINENSE

NEY PARANAGUÁ DE CARVALHO

**Relative size and *csw*-efficiency:
two new concepts based on *DEA* to guide
management of capital**

Tese de Doutorado submetida ao Programa de Pós-Graduação em Computação da Universidade Federal Fluminense como requisito parcial para a obtenção do título de Doutor. Área de concentração: Algoritmos e Otimização.

Orientador:
Carlos Alberto Martinhon

NITERÓI

2016

Relative size and *csw*-efficiency:
two new concepts based on *DEA* to guide management of capital

NEY PARANAGUÁ DE CARVALHO

Tese de Doutorado submetida ao Programa de Pós-Graduação em Computação da Universidade Federal Fluminense como requisito parcial para a obtenção do título de Doutor. Área de concentração: Algoritmos e Otimização.

Aprovada por:

Prof. Dr. Carlos Alberto Martinhon/ IC-UFF (Presidente)

Prof. Dr. Yuri Abitbol de Menezes Frota / IC-UFF

Profa. Dra. Simone de Lima Martins / IC-UFF

Prof. Dr. João Carlos C. B. Soares de Mello / Produção-UFF

Dra. Eliane Gonçalves Gomes / EMBRAPA

Prof. Dr. Roberto Quirino do Nascimento / DCOC-CI -UFPB

Niterói(RJ), 31 de março de 2016.

Dedico este trabalho a Alice Paranaguá e Sônia Paranaguá, respectivamente, minhas avó materna e mãe. A Vanessa, João Pedro e Maria Alice, meu monolito. Ao Jesualdo Filho, querido amigo.

Agradecimentos

Ao meu amigo e orientador Prof. Carlos Alberto de Jesus Martinhon, pelo acolhimento, orientação e comprometimento, presença indelével em todas as etapas deste trabalho;

A Profa. Mariana Rodrigues de Almeida, por seu apoio, sugestões e críticas;

Ao Instituto Federal de Tecnologia do Piauí e a Universidade Federal Fluminense, por terem proporcionado este DINTER;

Aos professores do programa de Pós-Graduação em Ciência da Computação da UFF que se dignaram em nos honrar com as suas aulas;

Aos meus irmãos, Cláudia, José Filho e Ricardo, aos cunhados Rogério e Cristiane, aos sobrinhos, Isabella, Victor, Rodrigo, Vinicius, Lucas e Alice, por serem uma família maravilhosa;

Ao amigo fraterno Adalton de Sena Almeida, repositório de toda a confiança de uma jornada de décadas, sempre juntos, em nome de quem agradeço a cada companheiro das empresas Infoway e Uniplam, que são profundamente amadas;

Ao amigo Prof. Ricardo de Andrade Lira, pela paciência e dedicação inestimáveis.

Ao aluno Anderson de Oliveira Farias, jovem talentoso que empenhou os seus esforços na implementação do *software* aplicativo desta pesquisa;

Aos amigos diletos, Eyder Franco Sousa Rios e Gilberto Farias de Sousa Filho, pelo carinho e atenção que me brindaram nas estadas em Niterói;

Ao amigo Fernando Pereira da Silva, presente em todas as horas.

Resumo

Um importante atributo das organizações é a sua **eficiência**, objeto da atenção de *Data Envelopment Analysis* (*DEA*), uma técnica não-paramétrica formulada através de um modelo de programação linear que, utilizando-se dos conceitos de *input* virtual e *output* virtual aplicados sobre dados observados, consegue comparar a eficiência entre organizações semelhantes. Por outro lado, o porte, um outro importante atributo das organizações, usualmente não tem sido calculado a partir de uma apreciação ampla dos aspectos que as compõem mas, tão somente a partir de um único aspecto escolhido para representá-las, o que pode ocasionar vieses. Com o objetivo de se obter uma medida mais justa para o porte e baseado em *DEA*, este trabalho propõe uma nova medida denominada **porte relativo**. Na sequência, novos modelos não lineares de programação matemática são propostos com o objetivo de estabelecer a melhor relação entre eficiência, porte relativo e uma certa disponibilidade (ou indisponibilidade) de **capital** (orçamento). Em adendo, as questões de supereficiência e sobreporte são endereçadas por meio de novos algoritmos. Ao final, estes conceitos são aplicados em um estudo de caso envolvendo os planos de saúde brasileiros.

Palavras-chave: Gestão de Capital, *csu*-eficiência, Porte Relativo, Supereficiência, Data Envelopment Analysis, Programação não-Linear, Planos de Saúde

Abstract

An important attribute of organizations is their **efficiency**, the object of attention of *Data Envelopment Analysis* (*DEA*), a nonparametric technique formulated through a linear programming model that, using the *virtual input* and *virtual output* concepts applied to observed data, is able to compare the efficiency itself among similar organizations. On the other hand, the size, another important attribute of organizations, have not usually been calculated from a broad appreciation of the aspects that make them, but solely from one aspect chosen for their representations, which can cause biases. In order to get a fairer measure of size and based on *DEA*, this paper proposes a new measure called **relative size**. Further, new nonlinear models of mathematical programming are proposed in order to establish the best relationship between efficiency, relative size and a certain availability (or unavailability) of **capital** (budget). In addendum, the *superefficiency* and *relative oversize* issues are addressed through new algorithms. Finally, these concepts are applied in a case study involving the Brazilian health plans.

Keywords: Capital Management, *csu*-efficiency, Relative Size, Superefficiency, Data Envelopment Analysis, Non-linear Programming, Health plans

Contents

List of Figures	x
List of Tables	xii
1 Introduction	1
1.1 Introduction	1
1.2 Motivation	2
1.3 Objectives	3
1.4 Conceptual Architecture	4
1.5 Document Structure	4
2 Data Envelopment Analysis	6
2.1 <i>DEA</i> Fractional Formulation	7
2.2 <i>DEA</i> Linear Formulation	8
2.2.1 Charnes-Cooper Transformation	8
2.2.2 CCR Primal	9
2.2.3 <i>CCR</i> Dual	9
2.3 <i>DEA</i> Strenghts	10
2.4 <i>DEA</i> Weaknesses	10
2.5 Comments	11
3 Relative size and ℓ_2-efficiency	12
3.1 ℓ_2 -efficiency and Relative Area	14

3.2	Relative Size and Range Area	16
3.3	Comments	20
4	Management of capital of single DMU	21
4.1	Minimization of capital	25
4.2	Maximization of relative size or csw -efficiency	26
4.3	Super-efficiency and Relative Oversize	28
4.4	Comments	32
5	Management of capital restricted to a subset of $DMUs$	33
5.1	Minimization of the Total Capital	34
5.2	Maximization of Global Efficiencies	36
5.3	Super-efficiency of one or more $DMUs$	37
5.4	Comments	42
6	Brazilian Health Plans Modeling	43
6.1	Phase 1 - DEA Modeling	44
6.1.1	Variables Selection	44
6.1.2	$DMUs$ Sample	47
6.1.3	DEA Model	48
6.1.4	Sample Assessment	48
6.2	Phase 2 - Relative sizes and ℓ_2 -efficiencies	51
6.3	Phase 3 - Management of Capital	51
6.3.1	Single DMU - Minimization of Capital	54
6.3.2	Single DMU - Maximization of Size	56
6.3.3	Single DMU - Maximization of Efficiency	59
6.3.4	$DMUs$ Subset - Minimization of Total Capital	61
6.3.5	$DMUs$ Subset - Maximization of Global Efficiency	63

6.3.6	<i>DMUs</i> Subset - Superefficiency	63
6.4	Comments	65
7	Conclusions and Future Directions	66
	References	69
	Appendix A - <i>DEACM</i> USER'S MANUAL	73
A.1	GENERAL INFORMATION	73
A.1.1	System Overview	73
A.2	Acronyms and Abbreviations	74
A.3	SYSTEM SUMMARY	74
A.3.1	Data Flows	74
A.4	GETTING STARTED	75
A.4.1	Data Input	76
A.4.1.1	Importing data	77
A.4.1.2	Adding an input	77
A.4.1.3	Adding an output	77
A.4.1.4	Adding an <i>DMU</i>	78
A.4.1.5	Deleting an input	78
A.4.1.6	Deleting an output	78
A.4.1.7	Deleting an <i>DMU</i>	78
A.4.1.8	Go to next step	78
A.5	<i>DEA</i>	78
A.5.1	<i>DMU</i> Details	79
A.5.2	Going Back	79
A.5.3	Go to next step	79
A.6	<i>KAO</i>	80

A.6.1	<i>DMU</i> Details	80
A.6.2	Going Back	81
A.6.3	Go to next step	81
A.7	Model Selection	81
A.7.1	Going Back	82
A.7.2	Go to next step	82
A.8	Parameters	82
A.9	Results	90

List of Figures

1.1	Conceptual Architecture	4
3.1	<i>Relative area</i> concept (a) <i>DEA efficiency</i> represented through $\tan \alpha_j$; (b) <i>Relative area</i> represented through <i>virtual input</i> and <i>virtual output</i>	12
3.2	Possible representations of the <i>Range area</i> (a) <i>Range area</i> represented through <i>virtual inputs with costs</i> and <i>virtual outputs</i> ; (b) <i>Range area</i> represented through relative sizes and ℓ_2 -efficiencies. We assume that $DMU_0 \equiv DMU_4$ is under investigation.	19
5.1	Movements towards the DEA solution. Subset $\Psi = \{DMU_3, DMU_4\}$. (a) Construction of DEA super-efficiencies (<i>red</i> points); (b) Repositioning all <i>DMU's</i> inside the <i>range area</i> (<i>yellow</i> points).	42
A.1	<i>DEACM</i> Steps	75
A.2	Data Input	76
A.3	Data Input	77
A.4	<i>DEA</i> Computation	79
A.5	<i>DMU</i> Details	80
A.6	<i>KAO</i> Computation	81
A.7	Model Selection	83
A.8	Single <i>DMU</i> - Capital Minimization	84
A.9	Single <i>DMU</i> - Maximation of Relative Size	85
A.10	Single <i>DMU</i> - Maximation of Efficiency	86
A.11	Subset of <i>DMUs</i> - Superefficiency Mode	87
A.12	Subset of <i>DMUs</i> - Altruistic or Selfish approaches	87
A.13	Subset of <i>DMUs</i> - Minimization of Capital	88

A.14 Subset of <i>DMUs</i> - Maximization of Efficiency	89
A.15 Result of the Minimization of Capital for a single target DMU	90
A.16 Result of the Maximization of Relative Size for a single target DMU	91
A.17 Result of the Maximization of Relative Efficiency for a single target DMU .	91
A.18 Result of the Minimization of Capital for <i>DMUs</i> subset	92
A.19 Result of Maximization of Efficiency for <i>DMUs</i> subset	92

List of Tables

3.1	Example with 5 <i>DMU</i> s. Computation of ℓ_2 -efficiencies, relative areas and relative size.	19
4.1	Definition of strategies in the management of capital. Parameters c_0 , s_0 and e_0 , respectively, denote the objective function value in the problems of <i>Minimization of Capital</i> , <i>Maximization of Relative Size</i> and <i>Maximization of Efficiency</i>	23
4.2	Management of Capital with $DMU_0 \equiv DMU_4$	31
5.1	Management of the Total Capital with $\Psi = \{DMU_3, DMU_4\}$. <i>Altruistic approach</i> with 5 <i>DMU</i> 's	41
6.1	Health plan <i>DEA</i> literature survey	44
6.2	Health plans attributes survey	45
6.3	Input health plans variables survey	46
6.4	Output health plans variables survey	47
6.5	Health Plans Data Sample	49
6.6	Efficiency Scores - <i>DMU</i> 's A to O	50
6.7	Efficiency Scores - <i>DMU</i> 's P to D1	50
6.8	Discriminant Analysis	50
6.9	Variables Correlation	51
6.10	Relative sizes and ℓ_2 -efficiencies	52
6.11	<i>DMU</i> s subsample for management of capital application	53
6.12	Minimization of Capital - $DMU_0 \equiv A$	54
6.13	Minimization of Capital - $DMU_0 \equiv C$	55
6.14	Minimization of Capital - $DMU_0 \equiv F$	55

6.15	Minimization of Capital - $DMU_0 \equiv C1$.	55
6.16	Minimization of Capital - $DMU_0 \equiv K$.	56
6.17	Minimization of Capital - $DMU_0 \equiv N$.	56
6.18	Maximization of Size - $DMU_0 \equiv A$.	57
6.19	Maximization of Size - $DMU_0 \equiv C$.	57
6.20	Maximization of Size - $DMU_0 \equiv F$.	57
6.21	Maximization of Size - $DMU_0 \equiv C1$.	58
6.22	Maximization of Size - $DMU_0 \equiv K$.	58
6.23	Maximization of Size - $DMU_0 \equiv N$.	58
6.24	Maximization of Efficiency - $DMU_0 \equiv A$.	59
6.25	Maximization of Efficiency - $DMU_0 \equiv C$.	60
6.26	Maximization of Efficiency - $DMU_0 \equiv F$.	60
6.27	Maximization of Efficiency- $DMU_0 \equiv C1$.	60
6.28	Maximization of Efficiency- $DMU_0 \equiv K$.	61
6.29	Maximization of Efficiency - $DMU_0 \equiv N$.	61
6.30	Minimization of Total Capital - Altruistic ($t_d = 1.1, t_j = 1, c = 90.59$)	62
6.31	Minimization of Total Capital - Selfish ($t_d = 1.1, t_j = 1, c = 0.63$)	62
6.32	Minimization of Total Capital - Altruistic ($t_d = 0.9, t_j = 1, c = -1.31$)	63
6.33	Minimization of Total Capital - Selfish ($t_d = 0.9, t_j = 1, c = -0.63$)	63
6.34	Maximization of Efficiency - Altruistic ($t_s = 1, t_j = 1, \bar{c} = 100$)	63
6.35	Maximization of Efficiency - Selfish ($t_s = 1, t_j = 1, \bar{c} = 100$)	64
6.36	Maximization of Efficiency - Altruistic ($t_s = 1, t_j = 1, \bar{c} = -100$)	64
6.37	Maximization of Efficiency - Selfish ($t_s = 1, t_j = 1, \bar{c} = -100$)	64
6.38	Superefficiency/ <i>ModelSE</i> - Altruistic ($t_d = 1.1, t_j = 1, c = -1.32$)	65
6.39	Superefficiency/ <i>ModelSE</i> - Selfish ($t_d = 1.1, t_j = 1, c = -1140$)	65

Chapter 1

Introduction

DEA (*Data Envelopment Analysis*) has been widely used since 1978 in order to determine relative efficiency of organizations. It provides an empiric-based method for modeling operational environment supported by a linear programming model, which avoids the determination of production functions, generally a hard task. However, all *DEA* scientific effort remain on the *efficiency* attribute. In this chapter is introduced a new possibility for *DEA* formulation, regarding other organization attribute, that is its *size*. Furthermore, both efficiency and size are combined in order to guide management of capital(budget) in organizations, which is the focus of present research.

1.1 Introduction

It would be very helpful if a user-friendly tool could estimate what kind of transformation would happen in an organization in the presence of some constraints of efficiency, size or capital (budget) or, even better, if that tool could be accurate about the most appropriate future operational scenario. These are common situations that require a lot of mental and emotional efforts from the decision maker, due to a lack of accessible objective informations that could, at least, somehow guide him. It is possible that, regarding to economic theory, some models can predict the impacts of management of capital in countries, states or in companies, as well as in [24] where Intel Corporation uses a combined framework with Martingale model of forecast evolution, stochastic programming, Monte Carlo simulation and isoprofit analysis to optimize capital investment in its supply chain. It seems to be very complex (and expensive) and not affordable, in general, to those small and medium-sized organizations. For these, there are few perspectives in being assured about determining jointed goals for capital, size and efficiency. In other words, there is not an

easy way to attach the appropriate goals to an available capital (or to be saved) or to efficiency or size rates, regarding the production characteristics of these organizations.

1.2 Motivation

From this point, a kind of tool to guide this management of capital must be supported by a reliable technique but, also, easy to model productive behavior issues, which obviously, does not demand the settlement of complex production functions. The major requisite for the right modeling would be decision maker's knowledge about his own organization and the market environment.

A technique that seems to be suited to these requisites is *Data Envelopment Analysis* (*DEA*), a fractional linear programming formulation proposed by Charnes, Cooper and Rhodes in 1978 [8] able to determine the efficiency of a production unit (or *DMU*, *Decision Making Unit*). It demands, mainly, well defined data, which is much more dependent to the knowledge of organization features, and avoids the determination of production functions.

The inner concept in *DEA* is to combine different inputs and outputs through weighted sums getting dimensionless virtual input and virtual output. The ratio from the latter to the former generates the efficiency concept (subject to constraints). There is an extensive *DEA* literature exploring many issues related to the efficiency itself and its attributes, e.g. frontier, *DMU* reference set, weights, slacks, etc. Thus, virtual input and output are an acceptable approximation of a organization production function, which is crucial to determine its operating system. From this perspective, virtual input and output could be explored in determining other organizational attributes, as size.

Organization size in literature is usually determined by a proxy variable, as it can be observed in flying companies load and passengers capacity [32]. It's easy to observe that the choice of just one variable may be an unfair measurement of organization's size which would lead to an inadequate analysis. For example, in the extreme, take an Information Technology Company exponentially profitable and valued with only few teammates against another regular one, with hundred of employees, but very far from the former in terms of profits and market value. In this case it's hard to determine the relative size between them, if either profits/value or number of employees are taking account. Through this research it is intended to formulate a new measure for organization size, which is broadly representative.

At last, since it is defined both efficiency and size, supported by *DEA*, this research aims to build mathematical programming models that look for determine exact arrangements of capital, efficiency and size appropriate to a specific organization and a given strategic decision.

1.3 Objectives

The objectives of the present research are defined as follows:

Main Objectives:

- develop a new formulation for organization size based on *DEA's* virtual input and virtual output concepts;
- formulate mathematical programming models for management of capital combined with efficiency and size, both based on *DEA*;
- apply the management of capital models to the Brazilian health plans area.

Specific Objectives:

- review of *DEA* bibliography;
- review of specific *DEA* common weights bibliography;
- formulate theoretical basis for a new size concept;
- develop mathematical programming models for a single *DMU*;
- develop mathematical programming models for a *DMUs* subset;
- survey of *DEA* bibliography applied to health area;
- select the appropriate variables in a *DEA* modeling for a health plans sector;
- build a *DMUs* set used in a *DEA* modeling for Brazilian health plans;
- apply the management of capital models to health plans sector;
- analyse outcomes consistency;
- build a software application in order to solve the management of capital models.

1.4 Conceptual Architecture

This work is based on a three-step conceptual architecture, structured as follows (Figure 1.1):

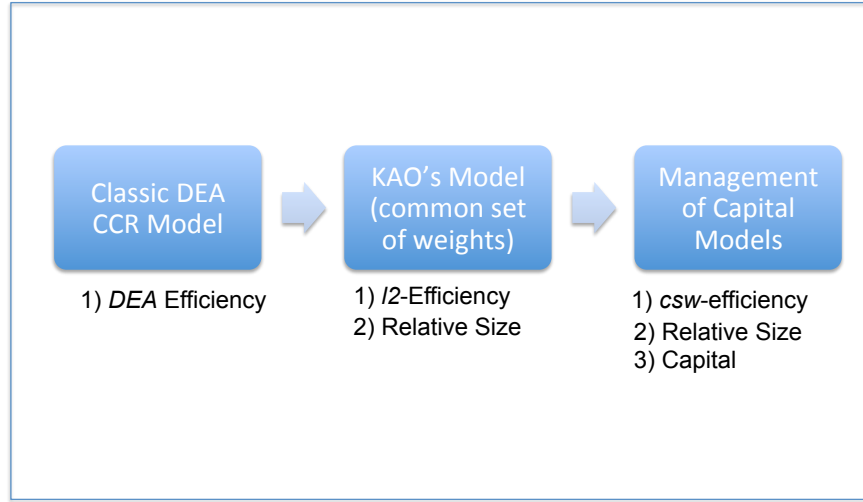


Figure 1.1: Conceptual Architecture

- the first step consists of using the classic *DEA CCR* input oriented technique to determine the efficiency scores which work as benchmarks in further steps;
- the second step involves the computing of common set of weights, relative sizes and ℓ_2 -efficiencies;
- the third step identifies the appropriate management of capital model and performs the specific instance modeling, defining goals for capital (budget), efficiency and size as well as setting the lower bounds for all input and output variables. At the end, *csw*-efficiencies and new relative sizes are all computed.

1.5 Document Structure

This document is organized as following:

- Chapter 2 presents the basic concepts of *Data Envelopment Analysis*;
- Chapter 3 presents theoretical bases for Relative size and ℓ_2 -efficiency concepts;
- Chapter 4 presents Management of Capital models for a single *DMU*;

-
- Chapter 5 presents Management of Capital models for a *DMUs* subset;
 - Chapter 6 presents an application to Brazilians health plans;
 - Conclusions presents a review of work results, observing positive aspects as well as the ones to be improved, and future directions are pointed;
 - Appendix presents the *DEACM* User's Guide, a software application for solving management of capital models.

Chapter 2

Data Envelopment Analysis

Data Envelopment Analysis (DEA) is a non-parametric technique to measure relative efficiency proposed by Charnes, Cooper and Rhodes in 1978 [8]. It extends the Farrel's work [9] in determining efficiency frontiers and it is able to handle multiple diverse inputs and outputs. It is used to identify the relative efficiency of a production unit under observation, called *Decision Making Unit (DMU)*, against a efficiency frontier built by itself and others *DMU's*, taking part of a reference set. The following requisites must be attained:

- *DMUs* set has to be homogeneous, so it must perform the same tasks and meet the same goals;
- *DMUs* have to operate under the same environment assumptions;
- Inputs and outputs parameters have to be the same for each *DMU*, varying only about their intensity and magnitude.

An important concept introduced by *DEA* is to combine inputs (X) and outputs (Y) linearly so they form the virtual input and output one. Doing this, the model is able to handle multiple inputs and outputs as dimensionless unique entities, which was a limitation to earlier Farrel's approach [9]. Thus, a virtual input is the sum of products of each input by its weight and a virtual output, in the same way, is the sum of products of each output by its own weight. The weights denote how each input and output contribute in composing the productive process.

2.1 DEA Fractional Formulation

In determination of relative efficiency, *DEA* aims to maximize the ratio of the virtual output (Y) by the virtual input (X), defined in Equation 2.1. The result of this process is the greatest efficiency that *DMU* in observation (DMU_0) can reach when it is compared to a frontier built by *DMUs* with the best practices. The value 1 in maximization process means that the DMU_0 is efficient when it is compared to others *DMUs*. On other hand, a value less than 1 means that DMU_0 is inefficient if compared to others *DMUs* of the set, which reach the ratio 1 before the DMU_0 , that is the condition to stop the model computation. Then, it is said that others *DMUs*, which reach the value 1, compounds the efficiency frontier and 'envelop', in other words, bound, those ones so called inefficient.

The fractional linear programming of *DEA* [8] is

$$e_0^* = \max \left(\frac{\sum_{r=1}^s u_r Y_{r0}}{\sum_{i=1}^m v_i X_{i0}} \right) \quad (2.1)$$

$$s.t. : \frac{\sum_{r=1}^s u_r Y_{rj}}{\sum_{i=1}^m v_i X_{ij}} \leq 1, \quad \forall j = 1, \dots, n. \quad (2.2)$$

$$u_r, v_i \geq 0, \quad \forall r = 1, \dots, s \text{ and } i = 1, \dots, m. \quad (2.3)$$

where:

- Y and X : are parameters that denote the *DMUs* observed outputs and inputs values;
- s , m and n : are the amount of outputs, inputs and *DMUs*, respectively;
- u and v : are variables that represent inputs and outputs weights;
- 2.1: is the objective function that determines the maximization of the ratio of virtual outputs by virtual inputs;
- 2.2: is the constraint that, by defining the possibilities allowed for variables u and v , set limits to the objective function maximization. Thus, the productive behavior of DMU_0 is referenced by the *DMUs* observation set, that bounds and determines the DMU_0 relative efficiency.

This formulation is called *CCR* and it provides constant returns to scale (*CRS*), which means that the virtual output changes proportionally along with the virtual input.

2.2 DEA Linear Formulation

Since the objective function 2.1 is a ratio and the variables u and v are superior unbounded, the global optimum value can be obtained from infinite optimal solutions. In other words, if (u^*, v^*) is an optimal solution, then $(\alpha u^*, \alpha v^*)$ is also optimal for $\alpha > 0$. In order to obtain a representative solution, the Charnes-Cooper transformation can be used to convert the fractional *DEA* into a linear form.

2.2.1 Charnes-Cooper Transformation

From *DEA* fractional model, the virtual input (objective function denominator) can be rewritten as it follows:

$$\sum_{i=1}^m v_i X_{i0} = \frac{1}{t} \quad (2.4)$$

and

$$\sum_{i=1}^m t v_i X_{i0} = 1 \quad (2.5)$$

Replacing 2.4 in objective function 2.1, we obtain:

$$\sum_{r=1}^s t u_r Y_{r0} \quad (2.6)$$

So, if we set:

$$\omega_i = t v_i \quad (2.7)$$

and

$$\mu_r = t u_r \quad (2.8)$$

And replacing u and v variables in 2.2:

$$\frac{\sum_{r=1}^s \frac{\mu_r}{t} Y_{rk}}{\sum_{i=1}^m \frac{\omega_i}{t} X_{ik}} \leq 1 \quad (2.9)$$

and

$$\sum_{r=1}^s \mu_r Y_{rk} - \sum_{i=1}^m \omega_i X_{ik} \leq 0 \quad (2.10)$$

Finally, the *DEA* fractional programming form can be rewritten in a linear one, taking 2.6, 2.5, 2.8, 2.7 and 2.10, as it follows:

$$e_0^* = \max \left(\sum_{r=1}^s \mu_r Y_{r0} \right) \quad (2.11)$$

$$s.t. : \sum_{i=1}^m \omega_i X_{i0} = 1, \quad (2.12)$$

$$\sum_{r=1}^s \mu_r Y_{rj} - \sum_{i=1}^m \omega_i X_{ij} \leq 0, \quad \forall j = 1, \dots, n. \quad (2.13)$$

$$\mu_r, \omega_i \geq 0, \quad \forall r = 1, \dots, s \text{ and } i = 1, \dots, m. \quad (2.14)$$

2.2.2 CCR Primal

The formulation above is called *CCR* primal input oriented form of *DEA* and its optimal value is the efficiency score of DMU_0 . This input oriented form aims to the input minimization while the opposite, the output orientation, aims to the output maximization and its optimal value is the transposed one obtained from the input orientation.

The *CCR* primal is also called multiplier form, since its solution are two vectors of weights that multiply the input and output values in order to produce the virtual output and input in the maximum ratio. Theses weights designate how input and outputs are combined in the production dynamics of DMU_0 .

2.2.3 CCR Dual

CCR dual form is called *the envelop form*, in which an efficiency frontier is built and it bounds, or envelops, DMU_0 . In dual form the constraints represent the inputs and

outputs, differently from the primal, in which the constraints represent the *DMUs* sample, which contains the *DMUs* benchmarks to DMU_0 and it can present just one *DMU*. The variables are: θ , which keeps the efficiency score; and λ , which points how inputs (or outputs) of a *DMU* from the reference set may be weighted in order to project a benchmark to DMU_0 , or how much the DMU_0 could reduce its inputs (or increase its outputs) to perform just as well as in the efficiency frontier.

The *CCR* dual input oriented formulation is as it follows:

$$e_0^* = \min \theta \quad (2.15)$$

$$s.t. : \sum_{k=1}^n \lambda_k X_{ik} - \theta X_{i0} \leq 0, \quad \forall i = 1, \dots, m. \quad (2.16)$$

$$\sum_{k=1}^n \lambda_k Y_{rk} - Y_{r0} \geq 0, \quad \forall r = 1, \dots, s. \quad (2.17)$$

$$\theta, \lambda_k \geq 0, \quad \forall k = 1, \dots, n. \quad (2.18)$$

2.3 *DEA Strenghts*

These are the most important aspects that support the use of *DEA*:

- Does not require a specific production function [25];
- Handles multiple different inputs and outputs, simultaneously [37];
- Can be employed even with small number of *DMUs* [30];
- Defines good practices and performance targets.

2.4 *DEA Weaknesses*

On the other hand, the major criticisms rely on:

- Specifying a model populated with good quality data is a very hard task when complex production processes are involved [25];
- Highly sensitiveness to outliers observations instead of average observations, since it is a data-driven technique;

- The efficiency definition is strongly dependent on the *DMUs* sample and it requires some complementary strategies to distinguish the statistical noise from the inefficiency [27] [19].

2.5 Comments

After the initial proposition, *DEA* has become object of great interest in the academic community and many extensions have been proposed so far. Some other models were proposed, as *BCC* (*Banker, Charnes and Cooper*) [4] that is highly used and it considers the scale efficiency introducing a new scalar variable in the linear programming model, or the new *Network DEA* [22] that decomposes a *DMU* in smaller functional parts, when they are connected as in a network, have their outputs used as inputs in the next production stage.

Liu *et al* 2013 [28] presents a survey about *DEA* application papers that demonstrates how this technique is spread from banking system, health sector, education or agriculture. Through this work we try to experiment its power in capital management issues.

Chapter 3

Relative size and ℓ_2 -efficiency

As seen previously, each instance of *DEA* on a *DMU* under observation (DMU_j) must lead to its relative efficiency, that is, how much it is efficient when it is compared to other *DMUs* in the sample. Thus, *DEA* is able to estimate the relative efficiency among all involved *DMUs* which is a very important attribute for the organizations. According to Figure 3.1.a, it can be assumed that $\tan \alpha = \frac{b_j}{a_j}$, where the coordinates (a_j, b_j) are virtual input and output, represents the relative efficiency of DMU_j .

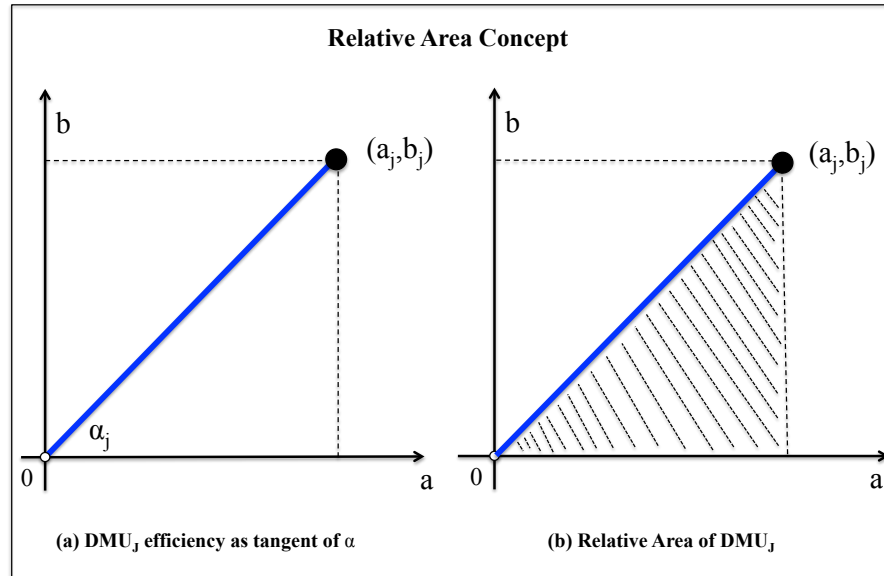


Figure 3.1: *Relative area concept (a) DEA efficiency represented through $\tan \alpha_j$; (b) Relative area represented through virtual input and virtual output.*

However, an interesting issue that could emerge in *DEA* are other observations over its graphical representation that could determine different attributes from the scores of relative efficiency. One could be interested, for instance, in the area below the segment P_1P_2 with $P_1 = (0,0)$ and $P_2 = (a_j, b_j)$ (Figure 3.1.b) that might be related to how large

is DMU_j operation (for some $j \in \{1, \dots, n\}$) compared to others $DMUs$ in the set. It seems to be an interesting view to be introduced, since in literature, only one variable is chosen as a proxy of operation size, which may bring bias in size determination. This comparative measure over DMU_j operation size could be named as *relative size* and it is derived from areas comparison (*relative area*). Since *DEA* tries to optimally combine different components (represented by input and output variables) of organizations, it sounds to be reasonable to determine relative sizes which are also based on their virtual inputs and outputs. Some hurdles however may be found in the direct application of the standard *DEA*. The $DMUs$ in *DEA* individually select the most advantageous weights in the computation of their efficiency scores which may prevent a fair comparison among the $DMUs$. As a consequence, the standard *DEA CCR* model is not exactly appropriated for dealing with this question. Note that any consistent measure of relative size, allowing a fair comparison among $DMUs$, should not consider weights dedicated to only one DMU . In order to overcome this hurdle, we can consider different models based on common set of weights which allow us to determine new efficiency scores (represented here by ℓ_2 -efficiencies) and define relative size on a more consistent basis. More specifically, in the work of Kao and Hung [23], the authors use the vector of *DEA* efficiencies as a target and propose different efficiency scores based on a common set of weights (the ℓ_p -efficiencies, for some integer $p \geq 1$). Based on the work of Yu [45], the authors in [23] deal with the ℓ_p -norm to compute the distance between a vector of ℓ_p -efficiencies and the target vector of *DEA* efficiencies. Some other works regarding common set of weights can be found in [12, 16, 34].

In this work, the framework proposed in [23] is used to deal with a common set of weights and define scores of *relative size*. In other words, based on the standard *DEA* technique and ℓ_p -efficiencies we hope to determine a comparative measure of how large is a DMU when it is compared to other $DMUs$ in the set. For this purpose, we fix $p = 2$ and deal with a quadratic programming problem to determine a vector of ℓ_2 -efficiencies. That will guarantee us uniqueness and Pareto's optimality¹ which allow us to consistently define new efficiencies and relative sizes on a common set of weights. Here, we will denote by \tilde{s}_j (for $0 < \tilde{s}_j \leq 1$), the *relative size* of DMU_j and we say that DMU_j is bigger than DMU_k , if and only if, $\tilde{s}_j > \tilde{s}_k$. The concepts of relative size and ℓ_2 -efficiency enable us to deal with different objectives and solve several problems related to *management of capital*² allowing controlled modifications of budget, relative size and efficiency scores.

¹We say that a solution is *Pareto efficient*, or *Pareto optimal*, if it is impossible to make any individual improvement without making at least one individual worse for any allocation of resources

²Here, we use the term "management of capital" to denote any economic activity carried out by one

In addition, we introduce the concept of *range area*, used in the joint representation of *csw*-efficiencies (more particularly, the ℓ_2 -efficiencies) and relative sizes. We say that a DMU_0 is inside the range area if its efficiency score and relative size are both positive and less or equal than *one*.

3.1 ℓ_2 -efficiency and Relative Area

In the standard *DEA CCR* Model, we consider a set of n *DMUs* and matrices X and Y with coordinates X_{ij} and Y_{rj} denoting, respectively, the i -th input (for $i = 1, \dots, m$) and the r -th output (for $r = 1, \dots, s$) of DMU_j (for $j = 1, \dots, n$). In this model, the *DMU* being considered selects the most advantageous weights subjected to constraints in which ratios of total weighted outputs to total weighted inputs of all *DMU*'s never exceed the unity (see chapter 2). In other words, *DEA* technique provides a measure of relative efficiency of a particular decision making unit (DMU_0) in order to compare it to a piece-wise efficiency frontier compound by *DMUs* of the reference set. Formally, without loss of generality, we can consider the following input-oriented *CCR* Model with costs associated to inputs which computes costs-based efficiency [10]. If we assume unitary costs then we have the standard input-oriented *CCR* Model as proposed in [8] (technical efficiency):

$$e_0^* = \max \left(\sum_{r=1}^s u_r Y_{r0} \right) \quad (3.1)$$

$$s.t. : \sum_{i=1}^m v_i A_{i0} X_{i0} = 1, \quad (3.2)$$

$$\sum_{r=1}^s u_r Y_{rj} - \sum_{i=1}^m v_i A_{i0} X_{ij} \leq 0, \quad \forall j = 1, \dots, n. \quad (3.3)$$

$$u_r, v_i \geq \epsilon > 0, \quad \forall r = 1, \dots, s \text{ and } i = 1, \dots, m. \quad (3.4)$$

with v_i and u_r denoting, respectively, the weights associated to the inputs and outputs. Further, vector (A_{10}, \dots, A_{m0}) represents positive costs of DMU_0 associated to inputs. The constant ϵ is a small non-Archimedean quantity to avoid unwanted zero weights. In the model (3.1)-(3.4), the goal is to maximize the efficiency score under the constraints in which efficiencies of all *DMU*'s are less or equal than 1. We denote by e_0^* the *DEA*-

or more *DMUs* involving saving or investment of resources.

efficiency of DMU_0 and say that DMU_0 is *efficient* whenever $e_0^* = 1$, otherwise, DMU_0 is *inefficient*.

Suppose that DMU_0 under investigation selects weights (u_r^0, v_i^0) for maximizing its efficiency score. We denote by $a_j^0 = \sum_{i=1}^m v_i^0 A_{ij}$ and $b_j^0 = \sum_{r=1}^s u_r^0 Y_{rj}$, respectively, the *virtual inputs with costs* (or just, *virtual inputs*) and *virtual outputs* of DMU_j related to DMU_0 . In addition, we say that $e_j^0 = b_j^0/a_j^0$ denotes the *efficiency of DMU_j related to DMU_0* . Note in the *DEA CCR Model* that if DMU_ℓ is under investigation then $e_k^\ell \leq e_k^*$ for $k = 1, \dots, n$.

As discussed earlier, the *DMUs* in *DEA* individually select the most advantageous weights in the computation of their efficiency scores, which may prevent a fair comparison among the *DMUs*. Therefore, if we are searching for a common set of weights, *DEA* efficiencies can be considered as target solutions for all *DMUs*. In other words, the idea is to find a common set of weights which produces efficiency scores that are closest to the target solution. In order to accomplish that, we can deal with the ideas described in [23] and [45]. After solving the *CCR Model* for each DMU_j , we build a target vector $e^* = (e_1^*, \dots, e_n^*)$ of *DEA* efficiencies and find an optimal common set of weights (\bar{u}, \bar{v}) to compute a new vector $e(\bar{u}, \bar{v}) = (e_1(\bar{u}, \bar{v}), \dots, e_n(\bar{u}, \bar{v}))$ of ℓ_p -efficiencies. In this case, the distance between $e(\bar{u}, \bar{v})$ and the target solution e^* is minimized. Obviously, we have $e_j(\bar{u}, \bar{v}) \leq e_j^*$, for $j = 1, \dots, n$. Making use of the ideas presented in Yu[45], the authors in [23] propose a model to compute a family of distance measures for integers $p \geq 1$. Here, for our purposes, we can consider the following optimization model:

$$\min D_p(e(u, v)) = \left(\sum_{j=1}^n (e_j^* - e_j(u, v))^p \right)^{1/p} \quad (3.5)$$

$$s.t. : e_j(u, v) = \frac{\sum_{r=1}^s u_r Y_{rj}}{\sum_{i=1}^m v_i A_{ij} X_{ij}} \leq 1, \quad \forall j = 1, \dots, n. \quad (3.6)$$

$$u_r, v_i \geq \epsilon > 0, \quad \forall r = 1, \dots, s \text{ and } i = 1, \dots, m. \quad (3.7)$$

where p represents the distance parameter and $A_{ij} > 0$ (for $i = 1, \dots, m$) denote costs associated to inputs. In the works of [23] and [45], the authors assume $A_{ij} = 1$ for $i = 1, \dots, m$. The objective function (3.5) minimizes the distance (denoted here by ℓ_p -distance) between a vector of ℓ_p -efficiencies and the target solution (vector of *DEA* efficiencies). In other words, function (3.5) guides the solution towards the *DEA* efficiency vector. Constraints (3.6) guarantee that ℓ_p -efficiencies are always less or equal than their corresponding *DEA* efficiencies and constraint (3.7) ensures positive weights.

In the model above, all deviations are equally weighted for $p = 1$ and it seems to be a natural choice in many applications. Note that as p increases, more weight is attributed to larger deviations. As discussed in [23], Pareto optimal solutions are attained for $1 \leq p < \infty$. In addition, optimal solutions for $p = 1$ and $p = \infty$ may not be unique which would cause difficulties in comparing the efficiency scores. In this case, a conflict might occur since two or more different optimal solutions may be associated to different efficiency scores (see [23] and [45]). Hence, in order to attain our objectives, the Euclidean distance obtained for $p = 2$, simultaneously guarantee us uniqueness and Pareto's optimality which is crucial for a consistent definition of relative size.

Further, the deviations of individual efficiency scores to their corresponding *DEA* efficiencies have the smallest variance and it is the best choice from the statistical point of view (see [23]). Thus, from now on, we only deal with the *Quadratic Problem* by assuming $p = 2$ and we will refer it as *ModelQP*. Note that *ModelQP* is a linearly constrained quadratic program and can be solved without difficulty by commercial NLP solvers.

Therefore, given a set of optimal weights (\bar{u}, \bar{v}) for *ModelQP*, we define $\bar{a}_j = \sum_{i=1}^s \bar{v}_i A_{ij} X_{ij}$ and $\bar{b}_j = \sum_{r=1}^s \bar{u}_r Y_{rj}$, respectively, the *virtual input with costs* and *virtual output* of DMU_j associated to (\bar{u}, \bar{v}) , for $j = 1, \dots, n$.

Now, we introduce the following definition:

Definition 1 (ℓ_2 -efficiency and Relative Area) Consider an optimal solution (\bar{u}, \bar{v}) for *ModelQP* above. Then, (i) ℓ_2 -**efficiency** of DMU_j is denoted by $\bar{e}_j = \bar{b}_j / \bar{a}_j$; and, (ii) **relative area** \bar{r}_j of DMU_j is denoted by $\bar{r}_j = (\bar{a}_j \bar{b}_j) / 2$, $\forall j = 1, \dots, n$. \square

We say that DMU_j is ℓ_2 -efficient whenever $e_j(\bar{u}, \bar{v}) = 1$, otherwise DMU_j is ℓ_2 -inefficient. Note that ℓ_2 -efficiencies are bounded on the $(0, 1]$ interval and $\bar{r}_j > 0$, for $j = 1, \dots, n$. Further, if (\bar{u}, \bar{v}) is optimal for *ModelQP* then all solutions $(k\bar{u}, k\bar{v})$ for $k > 0$ are also optimal for *ModelQP* (see [23]). This means that any decreasing or increasing of relative area (function of $k > 0$) does not affect the ℓ_2 -efficiencies and preserves the order of relative areas. Note that, if $\bar{r}_j > 0$ is associated to (\bar{u}, \bar{v}) then $\tilde{r}_j = k^2 \bar{r}_j$ is the new relative area associated to $(k\bar{u}, k\bar{v})$, for $k > 0$.

3.2 Relative Size and Range Area

Here, we will especially be concerned to the following problem. Assume that $\bar{e}_0 > 0$ and $\bar{r}_0 > 0$ denote, respectively, the ℓ_2 -efficiency and the relative area of DMU_0 . Further,

consider two parameters $k_e > 0$ and $k_s > 0$, denoting the updating rates of ℓ_2 -efficiency and relative size of the DMU_0 . Thus, if (\bar{a}_0, \bar{b}_0) represents the coordinates of DMU_0 , obtained from *ModelQP*, then we hope to determine new coordinates $(\tilde{a}_0, \tilde{b}_0)$ such that $\tilde{e}_0 = k_e \bar{e}_0$ and $\tilde{r}_0 = k_s \bar{r}_0$. The pair $(\tilde{e}_0, \tilde{r}_0)$ denotes, respectively, a new *csw*-efficiency and relative area of DMU_0 . Note that, every ℓ_2 -efficiency is also a *csw*-efficiency, however, the opposite is not true. Formally, the following result can be established:

Theorem 1 *Consider a DMU_0 with coordinates (\bar{a}_0, \bar{b}_0) . In addition, consider parameters $k_e > 0$ and $k_s > 0$ such that $\tilde{e}_0 = k_e \bar{e}_0$ and $\tilde{r}_0 = k_s \bar{r}_0$. Then, $\tilde{a}_0 = \bar{a}_0(k_s/k_e)^{1/2}$ and $\tilde{b}_0 = \bar{b}_0(k_s k_e)^{1/2}$.*

Proof: To prove this result two consecutive shifts of DMU_0 are combined. Firstly, the ℓ_2 -efficiency is preserved along with a displacement from (\bar{a}_0, \bar{b}_0) to (a'_0, b'_0) by assuming $k'_e = 1$ and $k'_s = k_s > 0$ (*translation movement*). Secondly, there is a movement from (a'_0, b'_0) to $(\tilde{a}_0, \tilde{b}_0)$, preserving the relative area by setting $k''_e = k_e > 0$ and $k''_s = 1$ (*rotation movement*).

Since $k'_e = 1$ and $k'_s > 0$ in the translation movement, thus $e'_0 = \bar{e}_0$ and $r'_0 = k'_s \bar{r}_0$. In addition, note that $e'_0 = b'_0/a'_0 = \bar{b}_0/\bar{a}_0$. Thus, from Definition 1 above is obtained that:

$$\frac{a'_0 b'_0}{2} = k'_s \frac{\bar{a}_0 \bar{b}_0}{2} \Rightarrow a'_0 = k_s \frac{\bar{a}_0 \bar{b}_0}{b'_0} \Rightarrow a'_0 = k_s \frac{\bar{a}_0 \bar{b}_0}{a'_0 e'_0} \Rightarrow (a'_0)^2 = k_s \frac{\bar{a}_0 \bar{b}_0}{e'_0}$$

Therefore, as $\bar{e}_0 = \bar{b}_0/\bar{a}_0$, is got:

$$a'_0 = \sqrt{\frac{k_s \bar{a}_0 \bar{b}_0}{\bar{e}_0}} \Rightarrow a'_0 = \bar{a}_0 \sqrt{k_s}. \quad (3.8)$$

Similarly,

$$\frac{a'_0 b'_0}{2} = k'_s \frac{\bar{a}_0 \bar{b}_0}{2} \Rightarrow b'_0 = k_s \frac{\bar{a}_0 \bar{b}_0}{a'_0} \Rightarrow b'_0 = \bar{b}_0 \sqrt{k_s} \quad (3.9)$$

which concludes the translation movement. Now, consider $r'_0 = \tilde{r}_0$ and set $k''_e = k_e > 0$ and $k''_s = 1$ to execute the rotation movement from (a'_0, b'_0) to $(\tilde{a}_0, \tilde{b}_0)$. Thus:

$$\frac{\tilde{a}_0 \tilde{b}_0}{2} = k''_s \frac{a'_0 b'_0}{2} \Rightarrow \tilde{a}_0 = \frac{a'_0 b'_0}{\tilde{b}_0} \quad (3.10)$$

Since $\tilde{e}_0 = k_e e'_0$, is obtained:

$$\frac{\tilde{b}_0}{\tilde{a}_0} = k_e \frac{b'_0}{a'_0} \Rightarrow \tilde{b}_0 = k_e \frac{b'_0 \tilde{a}_0}{a'_0} \quad (3.11)$$

After replacing, (3.11) and (3.8) in (3.10) can be concluded that:

$$(\tilde{a}_0)^2 = \frac{(a'_0)^2}{k_e} \Rightarrow \tilde{a}_0 = \frac{a'_0}{\sqrt{k_e}} \Rightarrow \tilde{a}_0 = \bar{a}_0 \sqrt{\frac{k_s}{k_e}} \quad (3.12)$$

Analogously, from (3.10) and (3.11), respectively, can be stated that:

$$\tilde{b}_0 = \frac{a'_0 b'_0}{\tilde{a}_0} \quad \text{and} \quad \tilde{a}_0 = \frac{a'_0 \tilde{b}_0}{k_e b'_0}. \quad (3.13)$$

Finally, since $b'_0 = \bar{b}_0 \sqrt{k_s}$, is obtained:

$$(\tilde{b}_0)^2 = (b'_0)^2 k_e \Rightarrow \tilde{b}_0 = \bar{b}_0 \sqrt{k_s k_e}$$

which proves the result. \square

Now, consider the following definition:

Definition 2 (*Relative Size and Normalization*) Let (\bar{a}_j, \bar{b}_j) be the coordinates of DMU_j associated to (\bar{u}, \bar{v}) , obtained from *ModelQP*, and let $\bar{r}_j = (\bar{a}_j \bar{b}_j)/2$ be their corresponding relative areas for $j = 1, \dots, n$. Then, $\tilde{s}_j = \bar{r}_j / \bar{r}$ is denoted the **relative size** of DMU_j where $\bar{r} = \max\{\bar{r}_j, \forall j = 1, \dots, n\}$. In addition, it is said that all DMU 's are **normalized** if their new coordinates $(\tilde{a}_j, \tilde{b}_j)$ are such that $\tilde{e}_j = \bar{e}_j$ and $\tilde{s}_j = (\tilde{a}_j \tilde{b}_j)/2$, for $j = 1, \dots, n$. \square

Note that $\tilde{s}_j \in (0, 1]$ for $j = 1, \dots, n$. Further, if $\tilde{s}_k = 1$ for some $k \in \{1, \dots, n\}$ then DMU_k has the *maximum relative size* in the set. Observe in the normalization above that $\tilde{e}_j = \bar{e}_j$ and $\tilde{s}_j = \bar{r}_j / \bar{r}$, for $j = 1, \dots, n$ and some $\alpha > 0$ (translation movement). Therefore, as an immediate consequence of Theorem 1 we have the following result:

Corollary 1 Consider (\bar{a}_j, \bar{b}_j) for $j = 1, \dots, n$, the coordinates of DMU_j associated to (\bar{u}, \bar{v}) (optimal solution of *ModelQP*). Then, $\tilde{a}_j = \bar{a}_j / \sqrt{\bar{r}}$ and $\tilde{b}_j = \bar{b}_j / \sqrt{\bar{r}}$ denote the new coordinates of DMU_j after normalization.

Proof: To prove that, it is sufficient to execute a translation movement of each DMU_j by setting $k_s = 1/\bar{r}$ and $k_e = 1$ in Theorem 1. \square

Observe in the *ModelQP* above that $D_2(e(\bar{u}, \bar{v})) = D_2(e(\tilde{u}, \tilde{v}))$ for $\tilde{u} = \bar{u} / \sqrt{\bar{r}}$ and $\tilde{v} = \bar{v} / \sqrt{\bar{r}}$. This means that the normalization in the computation of relative sizes (obtained

DMU_j	A_{1j}	X_{1j}	A_{2j}	X_{2j}	Y_{1j}	DEA efficiency	ℓ_2 -efficiency	\bar{r}_j	\tilde{s}_j
1	2	2	1	5	2	0.60	0.60	3.70	0.13
2	4	2	5	2	1	0.15	0.15	3.70	0.13
3	2	3	1	4	5	1	1	13.8	0.50
4	4	4	3	6	3	0.23	0.22	22.2	0.80
5	3	5	5	5	4	0.32	0.32	27.1	1

Table 3.1: Example with 5 $DMUs$. Computation of ℓ_2 -efficiencies, relative areas and relative size.

for $k_s = 1/\bar{r}$ and $k_e = 1$) does not affect the ℓ_2 -efficiencies and preserves the ordering of relative areas. To illustrate these concepts consider the following example with *five* $DMUs$, each of them with *two* inputs and *one* output as detailed in Table 3.1.

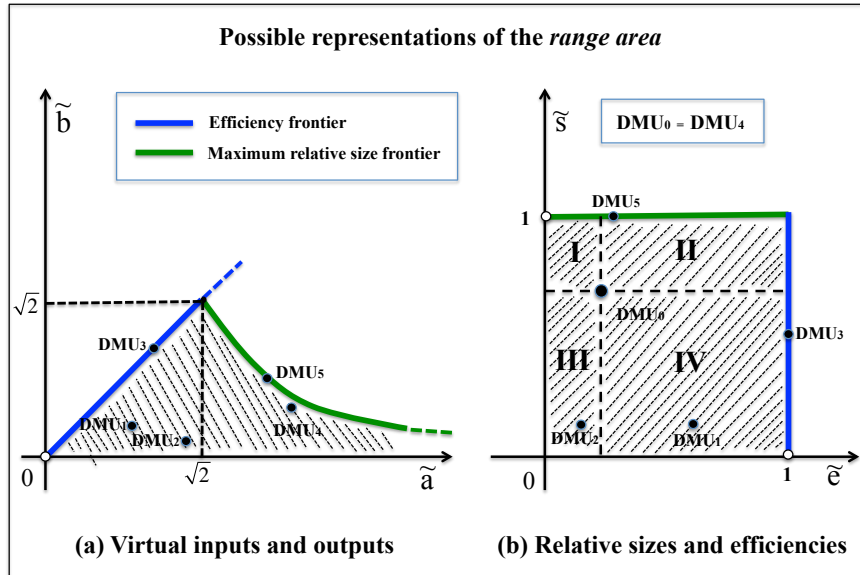


Figure 3.2: Possible representations of the *Range area* (a) *Range area* represented through *virtual inputs with costs and virtual outputs*; (b) *Range area* represented through relative sizes and ℓ_2 -efficiencies. We assume that $DMU_0 \equiv DMU_4$ is under investigation.

Note that virtual inputs with costs and virtual outputs of DMU_j can be represented, equivalently, by normalized coordinates $(\tilde{a}_j, \tilde{b}_j)$ (see the highlighted region of Figure 3.2.a). Further, the area below any segment P_1P_2 with $P_1 = (0, 0)$ and $P_2 = (\tilde{a}_j, \tilde{b}_j)$ denotes the relative size \tilde{s}_j of DMU_j and angle α_j (with $\tan(\alpha_j) = e_j(\tilde{u}, \tilde{v}) = \tilde{b}_j/\tilde{a}_j$), represents the ℓ_2 -efficiency of DMU_j . Here, we denote by *range area*, the set all coordinates $(\tilde{a}_j, \tilde{b}_j)$ corresponding to relative areas less or equal than *one* (relative sizes) and such that $0 < \tan(\alpha_j) \leq 1$, for $j = 1, \dots, n$. In addition, note in Figure 3.2.a that all DMU 's over the *green line* have maximum relative size while all DMU 's over the *blue line* are ℓ_2 -efficient. An equivalent representation with coordinates representing efficiency scores and relative sizes is detailed in Figure 3.2.b, which explicits displacements of the *range area* from a

$DMU_0 \equiv DMU_4$. Finally, for example, observe in Table 3.1 that DMU_3 is both DEA efficient and ℓ_2 -efficient while DMU_5 displays the maximum relative size.

3.3 Comments

Since we introduce the concepts of ℓ_2 -efficiency and relative size (along with range area), we are ready to show further how to combine them in the solution of a number of problems regarding management of capital. In this way, through the vector of ℓ_2 -efficiencies, new goals can be established so that new efficiency scores (*csu*-efficiencies) and relative sizes, inside the range area, are determined.

In order to achieve the goals, we use some linear and nonlinear objective functions and constraints in chapters 4 and 5, in which all proposed models can be solved by most available commercial NLP solvers. However, some of the models are non-convex and they can return a local optimum solution which may represent a drawback in some situations. To be more confident about the quality of the tested instances, a number of commercial solvers is used in order to compare different solution strategies (BARON 15.9.22, DONLP2, LOQO 6.02, MINOS 5.51, SNOPT 7.2 – 8 and LANCELOT). Despite the non-convexity, all of the tested models are well behaved and they produced the same optimal solution for all the tested instances.

Chapter 4

Management of capital of single *DMU*

In management of capital problems we assume that inputs associated to *DMUs* have different costs, and a manager may establish a number of objectives and directions motivated by different economic scenarios. Here, we propose a framework that jointly evaluate *three* parameters relating efficiency, capital and relative size. For instance, a manager can define new goals of relative size and/or efficiency in the constraints while minimizing the capital (or budget) of DMU_0 necessary to achieve that. In this case, quantities of individual items in the inputs should be eliminated, acquired or relocated. In such models, a *negative* (resp., *positive*) capital in the value of the objective function indicates that any saving (resp., investment) should be made in order to achieve the goals. For instance, in the case of a downturn of the economy, some decreasing of the relative size and/or efficiencies may be wanted trying to save the maximum amount of capital (“negative capital”). Another alternative is to inject a fixed amount of capital (“positive capital”) by purchasing or relocating some items in the inputs of DMU_0 while maximizing its relative size or efficiency.

After determining the vectors of ℓ_2 -efficiency and *relative size*, we are now ready to deal with a set of mathematical programming models which aims to guide the financial capital management on a particular *DMU*. These models are based on three parameters, *i.e.*, *capital* (or *budget*), *relative efficiency* and *relative size*. Thus, given a particular *DMU*, the idea is to optimize one of the three parameters subjected to a number of constraints. The other two parameters will be part of the constraints and will be fixed according to a strategy defined by the manager. In other words, minimization of capital, maximization of the efficiency score or maximization of size are three different problems addressed in the management of capital.

Thus, consider the range area as in chapter 3 and a DMU_0 under investigation with coordinates $(\tilde{a}_0, \tilde{b}_0)$, ℓ_2 -efficiency $\tilde{e}_0 = e_0(\tilde{u}, \tilde{v}) = \tilde{b}_0/\tilde{a}_0$ and relative size $\tilde{s}_0 = (\tilde{a}_0\tilde{b}_0)/2$. In order to clarify the possible strategies to be conducted by the manager, without loss of generality, turn back to the example of Figure 3.2 (page 19), assuming that $DMU_0 \equiv DMU_4$. Initially, note that we have a *one-to-one* correspondence between points in the range area of Figure 3.2.a and the unitary square of Figure 3.2.b. So, we can indistinctly denote the *range area* through the highlighted regions of Figures 3.2.a and 3.2.b, respectively. Additionally, observe in the Figure 3.2.b, the presence of *four* sectors denoting possible displacements of DMU_0 . In this way, any financial operation performed by DMU_0 will indicate removals or acquisitions of resources (inputs) in order to achieve the goal inside one of the sectors.

The strategy to be adopted by the manager is usually motivated by external factors and it will depend on the economic scenario. For instance, in the model of minimization of capital, to force a movement of DMU_0 towards a selected sector, the tight constraints of relative size and efficiency must be derived. A slowdown in the economy, for example, may result in a reduction of the relative size by the manager (to fit a decreasing demand) while keeping the same efficiency score. This corresponds to a new point between sectors *III* and *IV* (see Figure 3.2.b). In this case, the model of minimization of capital can indicate (through an updating of inputs X_{i0}), a negative budget showing the possibility of a better use of the available resources by DMU_0 . On the other hand, note that any point of sector *III* (even with negative budget) is not so interesting from the economic point of view since it represents a simultaneous reduction of efficiency and relative size.

A more ambitious movement should be a concurrent increasing of efficiency and relative size (sector *II*). If the model of capital minimization returns a positive value, new budget must be available in order to attain the goals, which shows some hardness to be achieved. Otherwise, a negative budget seems much harder and points out that a reduction of the available resources (inputs) must be implemented to attain the goals. Note that the solution indicated by the model may be regular, difficult or very difficult to achieve (see Table 4.1). In this case, a more appropriate and feasible movement could be adopted by the manager.

In present models, e'_0 and s'_0 denotes, respectively, the new *csw*-efficiency and relative size of DMU_0 inside the range area. Essentially, we want DMU_0 to move from $(\tilde{e}_0, \tilde{s}_0)$ to (e'_0, s'_0) . Thus, in the model of minimization of capital, given parameters $0 < k_e \leq 1/\tilde{e}_0$ and $0 < k_s \leq 1/\tilde{s}_0$, it is expected to determine new inputs and outputs in the constraints

OBJECTIVE	TARGET			AREA	HARDNESS
	Capital	Size	Efficiency		
Minimize Capital c_0	-	$s'_0 < \tilde{s}_0$	$e'_0 \geq \tilde{e}_0$	IV	High($\forall c_0$)
			$e'_0 < \tilde{e}_0$	III	Regular/Low($\forall c_0$)
	-	$s'_0 \geq \tilde{s}_0$	$e'_0 \geq \tilde{e}_0$	II	Very High($c_0 < 0$)
				II	High($c_0 \geq 0$)
			$e'_0 < \tilde{e}_0$	I	Regular/Low($\forall c_0$)
Maximize Size s_0	$\bar{c}_0 < 0$	-	$e'_0 \geq \tilde{e}_0$	IV	High($s_0 < \tilde{s}_0$)
				II	Very High($s_0 \geq \tilde{s}_0$)
			$e'_0 < \tilde{e}_0$	III	Regular/Low($s_0 < \tilde{s}_0$)
				I	Very High($s_0 \geq \tilde{s}_0$)
	$\bar{c}_0 \geq 0$	-	$e'_0 \geq \tilde{e}_0$	II	High($s_0 \geq \tilde{s}_0$)
				IV	Regular/Low($s_0 < \tilde{s}_0$)
			$e'_0 < \tilde{e}_0$	I,III	Regular/Low($\forall s_0$)
Maximize Efficiency e_0	$\bar{c}_0 < 0$	$s'_0 < \tilde{s}_0$	-	III	Regular/Low($e_0 < \tilde{e}_0$)
				IV	High($e_0 \geq \tilde{e}_0$)
		$s'_0 \geq \tilde{s}_0$	-	I	High($e_0 < \tilde{e}_0$)
				II	Very High($e_0 \geq \tilde{e}_0$)
	$\bar{c}_0 \geq 0$	$s'_0 < \tilde{s}_0$	-	III	Regular/Low($e_0 < \tilde{e}_0$)
				IV	High($e_0 \geq \tilde{e}_0$)
		$s'_0 \geq \tilde{s}_0$	-	II	High($e_0 \geq \tilde{e}_0$)
				I	Regular/Low($e_0 < \tilde{e}_0$)

Table 4.1: Definition of strategies in the management of capital. Parameters c_0 , s_0 and e_0 , respectively, denote the objective function value in the problems of *Minimization of Capital*, *Maximization of Relative Size* and *Maximization of Efficiency*.

such that $e'_0 = k_e \tilde{e}_0$ and $s'_0 = k_s \tilde{s}_0$. Therefore, if $1 < k_e \leq 1/\tilde{e}_0$ (resp., $0 < k_e < 1$) then some increasing (resp., some decreasing) of the efficiency score is wanted. Similarly, if $1 < k_s \leq 1/\tilde{s}_0$ (resp., $0 < k_s < 1$) then some increasing (resp., decreasing) of the relative size is wanted. Thus, the objective is to find the minimum capital c_0 necessary to attain goals e'_0 and s'_0 inside the range area (see Table 4.1). It is interesting to point out that it is possible to deal with the capital minimization model even for $k_e = k_s = 1$. In this case, a negative budget $c_0 < 0$ means that some rearrangement of entries (inputs) may be implemented to produce an economy of the available resources within the same production levels.

Now, for instance, suppose that a fixed capital $\bar{c}_0 > 0$ will be invested to attain a production (virtual output) with an improved efficiency $e'_0 = k_e \tilde{e}_0$, for some $k_e > 1$. In this case, it is searched to maximize the relative size of DMU_0 (represented by s_0 in Table 4.1) and simultaneously achieve e'_0 with budget $\bar{c}_0 > 0$ (sector *II*). Another possibility is to maximize the efficiency score (represented by e_0) assuming a fixed budget \bar{c}_0 to be saved (negative capital) and a new improved relative size equal to $s'_0 = k_s \tilde{s}_0$ for $k_s > 1$ (sectors *I* or *II*). In short, many possibilities of improvement can be investigated by the manager as summarized in the Table 4.1. The final decision on the choice of the better strategy will depend on the costs and technical difficulties involved in the operation.

As a final comment, note that any of the *three* optimization problems considered above can indicate a movement of DMU_0 out of the range area. It is said that DMU_0 is *super-efficient* if $e'_0 > 1$. Similarly, it is said that DMU_0 is *oversized* or has *relative oversize*, if $s'_0 > 1$. As a consequence, since all positions in the range area are relative, their corresponding efficiency scores (*csw*-efficiencies) and/or relative sizes must be updated. Later, in subsections 4.3 and 5.3, it is presented how to use Theorem 1 to deal with super-efficiency and relative oversize. In other words, it is displayed how to update the virtual inputs with costs and virtual outputs to determine new relative positions of all *DMUs* inside the range area.

In the sequence some auxiliary notation is defined. As discussed earlier, coefficients A_{ij} (for $i = 1, \dots, m$ and $j = 1, \dots, n$) denote the cost of input i associated to DMU_j . Thus, given matrices X_{ij} and Y_{ij} which are influenced by the cost of individual items A_{ij} , the idea is to determine matrices \bar{X}_{ij} and \bar{Y}_{ij} , representing new values of inputs and outputs. In order to accomplish that, we introduce variable δ_{ij} (resp., ϕ_{rj}) to represent the variation of input i (resp., output r) associated to DMU_j . In this way, constraints $\bar{X}_{ij} = X_{ij} + \delta_{ij} > 0$ and $\bar{Y}_{ij} = Y_{rj} + \phi_{rj} > 0$, respectively, will express new values for

inputs and outputs. It is important to remark that these constraints can be changed in real applications since more expensive items, for example, tend to be reduced first. In order to prevent essential items to be eliminated during the process, we can introduce specific bounds for both \bar{X}_{ij} and \bar{Y}_{ij} . Thus, based on specific aspects of each problem, we can assume $\delta_{ij} \geq -X_{ij} + L_{ij}$ and $\phi_{rj} \geq -Y_{rj} + U_{rj}$ for $L_{ij}, U_{rj} \geq \epsilon > 0$. Therefore, after solving *ModelQP* (chapter 3) to obtain normalized weights (\tilde{u}, \tilde{v}) and determine the efficiency scores and relative sizes of all *DMUs*, we describe the following models for the problem of management of capital of a single *DMU*.

4.1 Minimization of capital

As discussed earlier, in the problem of minimization of capital, a goal must be established by the manager according to a predetermined strategy. Essentially, given a DMU_0 under investigation with ℓ_2 -efficiency \tilde{e}_0 and relative size \tilde{s}_0 , the idea is to reach one of the *fours* sectors of the range area using the lowest possible capital (see Figure 3.2.b). To do that, positive constants k_e and k_s are set to compute e'_0 and s'_0 , respectively, the new efficiency score and new relative size of DMU_0 . As displayed in Table 4.1, combinations of k_e and k_s determine the new coordinates of DMU_0 . Formally, we establish the following model for the problem of minimization of capital, represented here by *ModelMC*:

$$c_0 = \min \left(\sum_{i=1}^m A_{i0} \delta_{i0} \right) \quad (4.1)$$

$$s.t. : \left(\sum_{i=1}^m \tilde{v}_i A_{i0} (X_{i0} + \delta_{i0}) \right) \left(\sum_{r=1}^s \tilde{u}_r (Y_{r0} + \phi_{r0}) \right) = 2k_s \tilde{s}_0; \quad (4.2)$$

$$\sum_{r=1}^s \tilde{u}_r (Y_{r0} + \phi_{r0}) - (k_e \tilde{e}_0) \sum_{i=1}^m \tilde{v}_i A_{i0} (X_{i0} + \delta_{i0}) = 0; \quad (4.3)$$

$$\delta_{i0} \geq -X_{i0} + L_{i0}; \quad \forall i = 1, \dots, m. \quad (4.4)$$

$$\phi_{r0} \geq -Y_{r0} + U_{r0}; \quad \forall r = 1, \dots, s. \quad (4.5)$$

The objective function (4.1) returns the value of the minimum capital c_0 considering the cost of the individual items assigned to DMU_0 . In the constraints (4.2) and (4.3), are updated both X_{i0} and Y_{r0} to compute the new relative size and efficiency of DMU_0 , respectively, s'_0 and e'_0 . Finally, constraints (4.4) and (4.5) define bounds imposed on the

variables to ensure that new inputs and outputs of DMU_0 (\bar{X}_{ij} and \bar{Y}_{ij}) are bigger than their corresponding lower bounds.

In addition, if $0 < k_e \leq 1/\tilde{e}_0$ and $0 < k_s \leq 1/\tilde{s}_0$, constraints (4.2) and (4.3) above guarantee that exactly one point (with coordinates (a'_0, b'_0)) belongs to one of the *four* sectors of the range area (see Figure 3.2.b). If these constraints are too restrictive, the manager can define larger areas so that the objective would be more easily attainable. For instance, we can change (4.2) and (4.3), respectively, by less restrictive constraints:

$$2k'_s \tilde{s}_0 \leq \left(\sum_{r=1}^s \tilde{u}_r (Y_{r0} + \phi_{r0}) \right) \left(\sum_{i=1}^m \tilde{v}_i A_{i0} (X_{i0} + \delta_{i0}) \right) \leq 2k''_s \tilde{s}_0 \quad (4.6)$$

and

$$k'_e \tilde{e}_0 \leq \frac{\sum_{r=1}^s \tilde{u}_r (Y_{r0} + \phi_{r0})}{\sum_{i=1}^m \tilde{v}_i A_{i0} (X_{i0} + \delta_{i0})} \leq k''_e \tilde{e}_0 \quad (4.7)$$

with $0 < k'_s < k''_s \leq 1/\tilde{s}_0$ and $0 < k'_e < k''_e \leq 1/\tilde{e}_0$. This is a more generic approach since there is no need the displacements to be restricted to one of the *four* sectors considered in the Figure 3.2.b.

In addition, note from constraints (4.2) and (4.3) above that, depending on the values of parameters k_e and k_s (if $k_e > 1/\tilde{e}_0$ or $k_s > 1/\tilde{s}_0$), we can found a pair (e'_0, s'_0) associated to coordinates (a'_0, b'_0) out of the range area. As mentioned earlier, since all efficiency scores and sizes are relative, we must update all relative positions $(\tilde{a}_j, \tilde{b}_j)$ for $j = 1, \dots, n$, to new coordinates inside the range area. Afterward in subsection 4.3, we show how to use Theorem 1 to solve this problem.

As a final comment, note that constraint (4.2) considered in the evaluation of the relative area is a bi-linear equality constraint and may produce a non-convex feasible region. Despite of that, as mentioned earlier, the problem seems to be well behaved for all tested solvers and it returned the same optimal solutions in all evaluated instances.

4.2 Maximization of relative size or *csw*-efficiency

In this subsection, consider two different objectives. Firstly, we maximize the relative size of DMU_0 , and secondly its efficiency score. In both cases, we assume that a fixed budget \bar{c}_0 (“negative” or “positive”) is settled beforehand. As discussed previously, a negative

(resp., positive) capital indicates that some saving (resp., investment) of the available resources is wanted to attain maximum relative sizes or *csw*-efficiencies. Therefore, in the first case, in order to maximize the relative size of DMU_0 we consider the following nonlinear programming formulation, represented here by *ModelMS*:

$$\max \left(\sum_{r=1}^s \tilde{u}_r(Y_{r0} + \phi_{r0}) \right) \left(\sum_{i=1}^m \tilde{v}_i A_{i0}(X_{i0} + \delta_{i0}) \right) / 2 \quad (4.8)$$

$$s.t. : \sum_{i=1}^m A_{i0} \delta_{i0} = \bar{c}_0; \quad (4.9)$$

$$\sum_{r=1}^s \tilde{u}_r(Y_{r0} + \phi_{r0}) - (k_e \bar{c}_0) \sum_{i=1}^m \tilde{v}_i A_{i0}(X_{i0} + \delta_{i0}) = 0; \quad (4.10)$$

$$\delta_{i0} \geq -X_{i0} + L_{i0}; \quad \forall i = 1, \dots, m. \quad (4.11)$$

$$\phi_{r0} \geq -Y_{r0} + U_{r0}; \quad \forall r = 1, \dots, s. \quad (4.12)$$

In the objective function (4.8), we maximize the value of relative size of DMU_0 . Constraint (4.9) shows how to update the inputs of DMU_0 with budget \bar{c}_0 . Constraint (4.10) guarantees a new efficiency score e'_0 with rate $k_e > 0$. Finally, constraints (4.11) and (4.12) impose lower bounds on the variables.

Now, in the second problem, we want to maximize the efficiency score of DMU_0 with budget \bar{c}_0 , it is sufficient to change the objective function (4.8) by function

$$e_0(\phi, \delta) = \frac{\sum_{r=1}^s \tilde{u}_r(Y_{r0} + \phi_{r0})}{\sum_{i=1}^m \tilde{v}_i A_{i0}(X_{i0} + \delta_{i0})} \quad (4.13)$$

and constraint (4.10) by

$$\left(\sum_{r=1}^s \tilde{u}_r(Y_{r0} + \phi_{r0}) \right) \left(\sum_{i=1}^m \tilde{v}_i A_{i0}(X_{i0} + \delta_{i0}) \right) = 2k_s \tilde{s}_0. \quad (4.14)$$

with $k_s > 0$ denoting the rate of the new relative size s'_0 . Here, *ModelME* denotes the problem of maximization of efficiency. Similarly to subsection 4.1, we assume that $0 < k_s \leq 1/\tilde{s}_0$ (for *ModelME*) and $0 < k_e \leq 1/\tilde{e}_0$ (for *ModelMS*). The situations where $k_s > 1/\tilde{s}_0$ and $k_e > 1/\tilde{e}_0$ will be discussed in the next subsection. In addition, note in both problems that the value of the objective function can also be bigger than 1. Thus, Theorem 1 can be used to ensure us new *csw*-efficiencies and relative sizes inside the range area.

4.3 Super-efficiency and Relative Oversize

As said earlier, new targets of efficiency and relative size may correspond to points which don not belong to range area (which features *super-efficiency* and *relative oversize*). In this case, *rotation* and *translation* movements are introduced here to recover the *csw*-efficiencies (not necessarily, ℓ_2 -efficiencies) and the relative sizes of all *DMUs* in the range area.

In this subsection, all the steps presented earlier in the management of capital are generalized and Theorem 1 is used to evaluate *two* new possible situations: namely, the *super-efficiency* and *relative oversize*. In order to simplify the discussion is only considered the problem of Minimization of Capital, represented here by *ModelMC*, and the other *two* cases are analogous.

Thus, after determining $(\tilde{e}_0, \tilde{s}_0)$ (resp., the ℓ_2 -efficiency and relative size of DMU_0), we can consider the updating rates $k_e, k_s > 0$ and solve *ModelMC* to obtain new coordinates (e'_0, s'_0) such that $e'_0 = k_e \tilde{e}_0 > 1$ (*super-efficiency*) and/or $s'_0 = k_s \tilde{s}_0 > 1$ (*relative oversize*). In this case, since all positions are relative and DMU_0 will be out of the range area, it is necessary to define a mechanism to update the efficiencies and relative sizes of all *DMUs*. In order to accomplish that, we deal with Theorem 1 that allows to determine new coordinates (inside the range area) through translation and rotation movements. In other words, Theorem 1 is used to correct the perturbation caused by DMU_0 .

In order to summarize the results, we can consider the following algorithm, divided in *two* parts. In the first part, from Step 01 to Step 07, weights (\tilde{u}, \tilde{v}) , ℓ_2 -efficiencies and relative sizes of all *DMUs* are determined. In the second part, from Step 08 to Step 12, the problem of management of capital is solved to obtain new *csw*-efficiencies and relative sizes.

Algorithm 1: { Minimization of Capital of DMU_0 . }

Inputs: Matrices X, Y, A ; DMU_0 ; lower bound matrices L and U , and parameters $k_e, k_s > 0$.

Outputs: Matrices \bar{X}, \bar{Y} ; capital c_0 , *csw*-efficiencies $e' = (e'_1, \dots, e'_n)$ and relative sizes $s' = (s'_1, \dots, s'_n)$ and coordinates (a'_j, b'_j) for $j = 1, \dots, n$.

Begin.

01. Compute the vector of DEA efficiencies $e^* = (e_1^*, \dots, e_n^*)$;

```

02.  $(\bar{u}, \bar{v}) \leftarrow \text{SolveModelQP}(X, Y, A, e^*);$ 
03. Compute virtual inputs (with costs) and outputs  $(\bar{a}_j, \bar{b}_j)$ , for  $j = 1, \dots, n$ ;
04. Compute:  $\tilde{e}_j \leftarrow \bar{b}_j/\bar{a}_j$  and  $\bar{r}_j \leftarrow (\bar{a}_j\bar{b}_j)/2$ , for  $j = 1, \dots, n$ ;
05.  $\bar{r} \leftarrow \max\{\bar{r}_j, \forall j = 1, \dots, n\}$ ;
06.  $\tilde{s}_j \leftarrow \bar{r}_j/\bar{r}$ , for  $j = 1, \dots, n$ ;
07. Determine the normalized weights  $\tilde{u} = \bar{u}/\sqrt{\bar{r}}$  and  $\tilde{v} = \bar{v}/\sqrt{\bar{r}}$ ;
08. Compute:  $\tilde{a}_j = \bar{a}_j/\sqrt{\bar{r}}$  and  $\tilde{b}_j = \bar{b}_j/\sqrt{\bar{r}}$  for  $j = 1, \dots, n$ ;
09.  $(c_0, \delta, \phi) \leftarrow \text{ExecuteModelMC}(X, Y, A, L, U, k_e, k_s, \tilde{u}, \tilde{v})$ ;
10.  $(\bar{X}, \bar{Y}) \leftarrow \text{UpdateXY}(\delta, \phi, X, Y)$ ;
11. Compute:  $\alpha_0 \leftarrow k_e \tilde{e}_0$  and  $\beta_0 \leftarrow k_s \tilde{s}_0$ ;
12. If  $((\alpha_0 > 1) \text{ or } (\beta_0 > 1))$  then {evaluate super-efficiency or oversize}
     $a \leftarrow \max\{1, \alpha_0\}$ ;
     $b \leftarrow \max\{1, \beta_0\}$ ;
     $k_e \leftarrow 1/a$  and  $k_s \leftarrow 1/b$ ;
    For  $j = 1, 2, \dots, n$  do {update all efficiencies and relative areas}
         $a'_j \leftarrow \tilde{a}_j(k_s/k_e)^{1/2}$ ;
         $b'_j \leftarrow \tilde{b}_j(k_s k_e)^{1/2}$ ;
         $e'_j \leftarrow b'_j/a'_j$ ;
         $s'_j \leftarrow (a'_j b'_j)/2$ ;
    end for;
    else {all DMU's are inside the range area}
         $e'_j \leftarrow \tilde{e}_j$ ; for  $j = 1, \dots, n$ ;
         $s'_j \leftarrow \tilde{s}_j$ ; for  $j = 1, \dots, n$ ;
    end if;
13. If  $\alpha_0 > 1$  then  $e'_0 \leftarrow 1$  else  $e'_0 \leftarrow \alpha_0$ ; {update position of  $DMU_0$ }
14. If  $\beta_0 > 1$  then  $s'_0 \leftarrow 1$  else  $s'_0 \leftarrow \beta_0$ ;
15. Return  $(c_0, e', s', a', b', \bar{X}, \bar{Y})$ ;
end.

```

In Step 1, through the *CCR*-Model, the vector e^* of DEA efficiencies is computed and used in the determination of a common set of weights (Step 2). From Steps 3 to 5, ℓ_2 -efficiencies, virtual inputs with costs and virtual outputs are computed and they will be used in the calculation of \bar{r} (maximum relative area). In Step 6 the relative size of all *DMU*'s is computed. In Step 7, using Corollary 1, the vector of common set of weights is normalized. In Step 8, the coordinates of all *DMU*'s are computed inside the range area.

In Step 9 (through *ModelMC*), the minimum capital c_0 and the changes on the inputs and outputs, represented by vectors δ and ϕ , respectively, are returned. In Step 10, δ and ϕ are used to calculate new matrices \bar{X} and \bar{Y} and in Step 11, the new efficiency score α_0 and relative size β_0 of DMU_0 are determined, along with the updating rates k_e and k_s . In Step 12, the possibility of *super-efficiency* and *relative oversize* is evaluated. Note that, if $\alpha_0 \leq 1$ and $\beta_0 \leq 1$ then the associated vector (a'_0, b'_0) is inside the range area, and only an updating of the coordinates of DMU_0 is performed. Otherwise, DMU_0 is out of the range area and it will affect the relative position of all DMU 's in the set. Thus, in order to reevaluate their *csw*-efficiencies into the range area, Theorem 1 is used to determine new coordinates (a'_j, b'_j) , for $j = 1, \dots, n$. In Steps 13 and 14 the displacement of DMU_0 inside the range area is computed. Finally, in Step 15, the updated matrices \bar{X} and \bar{Y} , capital c_0 to be saved/invested, vector $e' = (e'_0, \dots, e'_n)$ with new *csw*-efficiencies and vector $s' = (s'_0, \dots, s'_n)$ of relative sizes are returned.

Note that, after substituting $\tilde{a}_j = \sum_{i=1}^m \tilde{v}_i A_{i0} X_{ij}$ and $\tilde{b}_j = \sum_{r=1}^s \tilde{u}_r Y_{rj}$ in equations $a'_j = \tilde{a}_j (k_s/k_e)^{1/2}$ and $b'_j = \tilde{b}_j (k_s k_e)^{1/2}$, respectively, it is easy to identify new weights:

$$v'_i = \tilde{v}_i \sqrt{\frac{k_s}{k_e}} \quad \text{and} \quad u'_r = \tilde{u}_r \sqrt{k_s k_e} \quad \text{for } i = 1, \dots, m. \quad (4.15)$$

Assuming the same cost matrix A in the management of capital, it is important to remark that after computing new matrices \bar{X} and \bar{Y} , another set of optimal weights can be obtained, say (u_r^*, v_i^*) , which is determined through a new execution of *ModelQP*. Therefore, even if the coordinates (u'_r, v'_i) define a feasible solution with a common set of weights, they do not represent an optimal solution to this new problem. Note that, the *csw*-efficiencies and the relative sizes of all DMU s were determined by the perturbation caused by moving DMU_0 out of the range area and the weights (u'_r, v'_i) were adjusted to reevaluate their relative positions. Thus, a new execution of *ModelQP* with matrices \bar{X} and \bar{Y} and new *DEA* efficiencies may return a different efficiency score (ℓ_2 -efficiency) for DMU_0 (see the Example 1). Hence, an interesting question to investigate is the quality of the decision taken by DMU_0 after a new execution of *ModelQP*.

Example 1: In order to illustrate the discussions presented in subsections 4.1, 4.2 and 4.3 respectively, return to the example of Table 3.1. Initially, some parameters must be set regarding lower bounds for inputs and outputs ($L_{10} = 3, L_{20} = 4, U_{10} = 2$). Firstly, consider the model of minimization of capital and a hypothetical scenario where the managers of a company want to define an emergency plan motivated by some undesirable

external factor, for instance, due to an unexpected growth of demand. In this context, we suppose they want to evaluate the minimum budget necessary for doubling size (to meet the growing demand) allowing a reduction of 10% in the efficiency score. Remind that $DMU_0 \equiv DMU_4$ in Table 3.1 with coordinates $(\tilde{e}_4 = 0.22, \tilde{s}_4 = 0.80)$. In this case, $k_e = 0,9$ and $k_s = 2$ in Algorithm 1 must be set.

Thus, after solving *ModelMC*, the minimum budget ($c_0 = \$2.12$) is obtained, as presented in Table 4.2. Further, as $\beta_0 > 1$, the new coordinates (α_0, β_0) are out of the range area, which configures a relative oversize situation (see Step 12). Therefore, the execution of Algorithm 1 determine new coordinates (e'_j, s'_j) inside the range area for $j = 1, \dots, n$ (represented from columns 5 to 14). The objective function value of each model is denoted in bold.

In a second moment, according to subsection 4.2, assume that DMU_4 access a positive budget \bar{c}_0 equal to \$2.12 monetary units, the same value found in previous minimization of capital. Then, it is expected to obtain the maximum improvement of its relative size while decreasing its efficiency score by 10%. In this scenario, *ModelMS* points exactly the same value for relative size that was used as target in previous model ($\beta_0 = 1.60$), once again, reaching the same coordinates (α_0, β_0) , out of the range area. A precise correspondence between *ModelMC* and *ModelMS* is observed (Table 4.2).

Models	c_0	α_0	β_0	DMU_1		DMU_2		DMU_3		DMU_4		DMU_5	
				e'_1	s'_1	e'_2	s'_2	e'_3	s'_3	e'_4	s'_4	e'_5	s'_5
<i>ModelMC</i>	2.12	0.20	1.60	0.60	0.08	0.15	0.08	1.00	0.31	0.20	1.00	0.32	0.63
<i>ModelMS</i>	2.12(\bar{c}_0)	0.20	1.60	0.60	0.08	0.15	0.08	1.00	0.31	0.20	1.00	0.32	0.63
<i>ModelME₁</i>	2.12(\bar{c}_0)	0.79	1.60	0.60	0.08	0.15	0.08	1.00	0.31	0.79	1.00	0.32	0.63
<i>ModelME₂</i>	10(\bar{c}_0)	0.79	1.60	0.60	0.08	0.15	0.08	1.00	0.31	0.79	1.00	0.32	0.63

Table 4.2: Management of Capital with $DMU_0 \equiv DMU_4$.

In the *ModelME* will be considered *two* different instances, represented respectively, by *ModelME₁* and *ModelME₂*. In *ModelME₁*, it is intended to maximize the efficiency score of DMU_0 with budget $\bar{c}_0 = \$2.12$ and doubling its relative size. An expressive growth of efficiency score can be observed ($\alpha_0 = 0.79$), denoting that DMU_0 can reach such efficiency by attaining these goals. Once again coordinates (α_0, β_0) display themselves out of range area, featuring a relative oversize situation. Finally, in the row *ModelME₂*, assume that a bigger budget ($\bar{c}_0 = \$10$) must be used by DMU_0 while keeping the same relative size ($\beta_0 = 1.60$). In this case, despite the fact of higher capital availability to attain the target of relative size, the efficiency score reaches the same rate ($\alpha_0 = 0.79$). In short, a number of different simulations may help in the definition of a strategy to be adopted by the manager. \square

4.4 Comments

Models *ModelMC* and *ModelMS* show consistent behaviors, since there are correspondences among capital (budget) and relative size, not mattering they are objective functions or parameters goals in different models. Differently, in *ModelME* the use of capital and relative size parameters with values obtained from *ModelMC* and *ModelMS* previous computations, do not lead to the same values for relative efficiency as those which were set as parameters previously. That happens because the direction of the model is *maximization*, which takes the efficiency to its highest value attainable in the presence of capital and relative sizes goals. Sometimes, however, such situation is not achievable against the hard task that is to operate in higher levels of efficiency. On the other hand, the use of *minimization* direction to the model seems to be desirable in some situations, since one could decide to run its production unit in lowest efficiency level enough to reach capital and relative size goal. In this case, the new minimization of efficiency model presents complete correspondence to minimization of capital and maximization of size models.

The *DMUs* obtained from management of capital models may lead to new benchmarks in a next round of *DEA's* efficiency computing.

In the following chapter, the minimization of capital and the maximization of efficiency models are extended to manage a subset of *DMUs*, considering different approaches.

Chapter 5

Management of capital restricted to a subset of $DMUs$

Similarly to the management of capital imposed to a particular DMU , this approach can be extended in such way that can collectively determine new goals of performance restricted to a subset Ψ of $DMUs$. In order to guide this analysis, the Euclidean distance between vectors of ℓ_2 -efficiencies and DEA efficiencies is dealt and this distance is used to find policies of capital distribution. In this case, the capital (negative or positive) destined to all $DMUs$ of Ψ must be distributed and used to update their corresponding items in the inputs.

After computing the optimal vector of ℓ_2 -efficiencies, we establish some objectives that collectively determine new consumption input targets for a subset Ψ of $DMUs$. A possible objective would be the minimization of the total capital destined to the subset. For example, suppose that a manager wants to decrease by 10% the distance between the vectors of ℓ_2 -efficiencies and DEA efficiencies restricted to Ψ (this is equivalent to increase further their efficiency scores towards the DEA efficiencies). In this way, the objective would be the minimization of the total capital along with the establishment of the policies of capital distribution among the $DMUs$. As a consequence, capital c_j (or budget) destined to DMU_j could be used to eliminate, acquire or relocate its corresponding items in the inputs. Another objective would be the inputs modification of all $DMUs$ by fixing a total budget (positive or negative) and minimizing the distance between the vectors of DEA efficiencies and ℓ_2 -efficiencies. In other words, this is equivalent to maximize the efficiency scores of all $DMUs$ of Ψ dealing with a fixed budget necessary to accomplish that.

In next subsections two main objectives are presented: *Minimization of the Total*

Capital and Maximization of Global Efficiency. In both cases, only movements inside the range area are allowed. Additionally, two main approaches will be considered: *i)* the *altruistic approach* in which the *DMUs* of Ψ help the targets of the entire set of *DMUs*; and *ii)*, the *selfish approach* in which the *DMUs* of Ψ cooperate among themselves to help their own targets. Finally, in subsection 5.3, it is displayed how to use Theorem 1 to extend these *two* problems and how to deal with displacements out of the range area.

In this chapter the proposed models are slightly more difficult than those ones proposed in chapter 4. In addition to the bi-linear constraint associated to relative area, we also include some nonlinear fractional constraints. Despite of that, similarly to chapter 4, all tested instances produced the same optimal solution (local or global) for all tested solvers.

5.1 Minimization of the Total Capital

In the problem of *Minimization of the Total Capital*, we assume that new efficiency scores and relative sizes are settled for all *DMUs* of Ψ and the objective is to minimize the total capital c_0 necessary to achieve that. As mentioned above, negative (resp., positive) capital in the solution indicates that any saving (resp., investment) must be made in order to attain the goals.

In order to present these models, both on altruistic and selfish approaches, consider the normalized vectors (\tilde{u}, \tilde{v}) , which are the optimal common set of weights of *ModelQP*. In addition, assume $\tilde{D} \equiv D_2(e(\tilde{u}, \tilde{v}))$ and values \tilde{e}_j and \tilde{s}_j , respectively, the ℓ_2 -efficiency and relative size of *DMU_j* for $j = 1, \dots, n$. In addition, consider a parameter $t_d > 0$, the expected updating rate of the Euclidean distance between $e(\tilde{u}, \tilde{v})$ and e^* (resp., vectors of *csu*-efficiencies and *DEA* efficiencies). Finally, set $0 < t_j \leq 1$ as the parameters used in the definition of lower bounds for the relative sizes of every *DMU_j* of Ψ . Then, the following formulation can be established and denoted here by *ModelMTC₁*, for the problem of *Minimization of the Total Capital* on the altruistic approach:

$$c_0 = \min \left(\sum_{j \in \Psi} \sum_{i=1}^m A_{ij} \delta_{ij} \right) \quad (5.1)$$

$$s.t. : \left(\sum_{j=1}^n \left(e_j^* - \frac{\sum_{r=1}^s \tilde{u}_r (Y_{rj} + \phi_{rj})}{\sum_{i=1}^m \tilde{v}_i A_{ij} (X_{ij} + \delta_{ij})} \right)^2 \right)^{1/2} = t_d \tilde{D}; \quad (5.2)$$

$$2t_j \tilde{s}_j \leq \left(\sum_{r=1}^s \tilde{u}_r (Y_{rj} + \phi_{rj}) \right) \left(\sum_{i=1}^m \tilde{v}_i A_{ij} (X_{ij} + \delta_{ij}) \right) \leq 2; \quad \forall j \in \Psi. \quad (5.3)$$

$$\sum_{r=1}^s \tilde{u}_r (Y_{rj} + \phi_{rj}) - \sum_{i=1}^m \tilde{v}_i A_{ij} (X_{ij} + \delta_{ij}) \leq 0; \quad \forall j \in \Psi. \quad (5.4)$$

$$\delta_{ij} \geq -X_{ij} + L_{ij}; \quad \forall i = 1, \dots, m \text{ and } j \in \Psi. \quad (5.5)$$

$$\phi_{rj} \geq -Y_{rj} + U_{rj}; \quad \forall r = 1, \dots, s \text{ and } j \in \Psi. \quad (5.6)$$

$$\delta_{ij} = 0 \text{ and } \phi_{rj} = 0; \quad \forall i = 1, \dots, m, r = 1, \dots, s \text{ and } j \notin \Psi. \quad (5.7)$$

The objective function (5.1) returns the value of the minimum capital c_0 and computes the budget $c_j = \sum_{i=1}^m A_{ij} \delta_{ij}$ destined to each DMU_j of Ψ . In the constraint (5.2), a new vector of efficiencies is determined (the ratio of virtual outputs to inputs) in which the Euclidean distance from the target solution is exactly $t_d \tilde{D}$. A parameter $t_d > 1$ (resp., $t_d < 1$) means that a reduction (resp., an improvement) of the total efficiency is required. Note that, through parameter t_d , new efficiency goals were established for the entire set of $DMUs$ (altruistic approach). Constraint (5.3) indicates that new relative sizes varies between $t_j \tilde{s}_j$ and 1 for each DMU_j of Ψ with $0 < t_j \leq 1$. Constraint (5.4) ensures that new csw -efficiencies do not exceed the unity. Note that DEA efficiencies e_j^* can be exceeded, associated to matrices X and Y . However, the new DEA efficiencies (less or equal than *one*), associated to matrices \bar{X} and \bar{Y} , are never exceeded. Finally, constraints (5.5), (5.6) and (5.7) guarantee new matrices \bar{X} and \bar{Y} with coefficients bigger than their corresponding lower bounds L and U (established by the manager). Observe that only inputs and outputs associated to $DMUs$ of Ψ are updated.

Note in the proposed model, due to constraint (5.2), that only the changes in the matrices X and Y associated to $DMUs$ of Ψ will help, collectively, to reach the efficiency goals of all n $DMUs$ in the set. In order to attain this objective, the efficiency of some $DMUs$ of Ψ (*altruistic* approach) may even worsen. Another possibility is to pursue the goals only restricted to $DMUs$ of Ψ (*selfish* approach). In order to accomplish that,

constraint (5.7) is deleted and defined

$$\tilde{D}_\Psi = \left(\sum_{j \in \Psi} \left(e_j^* - \frac{\sum_{r=1}^s \tilde{u}_r Y_{rj}}{\sum_{i=1}^m \tilde{v}_i A_{ij} X_{ij}} \right)^2 \right)^{1/2}. \quad (5.8)$$

After that, we substitute constraint (5.2) by

$$\left(\sum_{j \in \Psi} \left(e_j^* - \frac{\sum_{r=1}^s \tilde{u}_r (Y_{rj} + \phi_{rj})}{\sum_{i=1}^m \tilde{v}_i A_{ij} (X_{ij} + \delta_{ij})} \right)^2 \right)^{1/2} = t_d \tilde{D}_\Psi. \quad (5.9)$$

Here, we represent the selfish approach by *ModelMTC*₂.

5.2 Maximization of Global Efficiencies

In the problem of *Maximization of Global Efficiencies*, given a fixed amount of capital \bar{c}_0 (positive or negative) and specific lower and upper bounds on the relative sizes, we expect to define a mechanism to distribute the capital \bar{c}_0 among the *DMUs* of Ψ in order to maximize their efficiency scores. On the altruistic approach, we minimize the Euclidean distance between the n -dimensional vector of *csw*-efficiencies and the *DEA* efficiencies. Similarly, on the selfish approach, we minimize the Euclidean distance only restricted to the coordinates of Ψ .

Formally, we have the following formulation on the altruistic approach, represented here by *ModelMGE*₁:

$$\min \left(\sum_{j=1}^n \left(e_j^* - \frac{\sum_{r=1}^s \tilde{u}_r (Y_{rj} + \phi_{rj})}{\sum_{i=1}^m \tilde{v}_i A_{ij} (X_{ij} + \delta_{ij})} \right)^2 \right)^{1/2} \quad (5.10)$$

$$s.t. : \sum_{r=1}^s \tilde{u}_r (Y_{rj} + \phi_{rj}) - \sum_{i=1}^m \tilde{v}_i A_{ij} (X_{ij} + \delta_{ij}) \leq 0; \quad \forall j \in \Psi. \quad (5.11)$$

$$\sum_{j \in \Psi} \left(\sum_{i=1}^m A_{ij} \delta_{ij} \right) = \bar{c}_0; \quad (5.12)$$

$$2t_j \tilde{s}_j \leq \left(\sum_{r=1}^s \tilde{u}_r (Y_{rj} + \phi_{rj}) \right) \left(\sum_{i=1}^m \tilde{v}_i A_{ij} (X_{ij} + \delta_{ij}) \right) \leq 2; \quad \forall j \in \Psi. \quad (5.13)$$

$$\delta_{ij} \geq -X_{ij} + L_{ij}; \quad \forall i = 1, \dots, m \text{ and } j \in \Psi. \quad (5.14)$$

$$\phi_{rj} \geq -Y_{rj} + U_{rj}; \quad \forall r = 1, \dots, s \text{ and } j \in \Psi. \quad (5.15)$$

$$\delta_{ij} = 0 \text{ and } \phi_{rj} = 0; \quad \forall i = 1, \dots, m, r = 1, \dots, s \text{ and } j \notin \Psi. \quad (5.16)$$

The objective function (5.10) denotes the distance between the n -dimensional vector of efficiencies (with common set of weights) and the vector of *DEA* efficiencies. Inequality (5.11) means that the new efficiency scores do not exceed the unity. Equality (5.12) computes the changes on the inputs and determines the budget $c_j = \sum_{i=1}^m A_{ij}\delta_{ij}$ assigned to each DMU_j of Ψ . Constraint (5.13), for $0 < t_j \leq 1$, determines relative sizes varying between their corresponding lower and upper bounds. Finally, (5.14), (5.15) and (5.16) are constraints imposed on the variables. Note that all changes in the inputs and outputs associated to *DMUs* of Ψ will help in the improvement of all n *csw*-efficiencies towards their corresponding *DEA* efficiencies.

Similarly, on the selfish approach (denoted here by *ModelMGE₂*), it is sufficient to delete constraint (5.16) and substitute objective function (5.10) by

$$\min \left(\sum_{j \in \Psi} \left(e_j^* - \frac{\sum_{r=1}^s \tilde{u}_r (Y_{rj} + \phi_{rj})}{\sum_{i=1}^m \tilde{v}_i A_{ij} (X_{ij} + \delta_{ij})} \right)^2 \right)^{1/2}. \quad (5.17)$$

As a final comment, note on the selfish approach that for $|\Psi| = 1$, the movement of the *DMU* under investigation is only towards its *DEA* solution. This procedure differs slightly from that of chapter 4 in which some displacements of DMU_0 out of the range area are allowed.

In the next section, consider the possibility of super-efficiency of one or more *DMUs* (both on altruistic and selfish approaches). This will allow that even efficient *DMUs* can increase further their efficiency scores.

5.3 Super-efficiency of one or more *DMUs*

Consider a subset Ψ of *DMUs* and each one with its corresponding ℓ_2 -efficiency and relative size. In addition, suppose that *DMUs* want to collaborate with each other (selfishly or not) in order to update their efficiencies scores and collectively achieve new goals in the constraints. Here, the idea is to allow that one or more *DMUs* of Ψ can escape from the range area. IN order to accomplish that, a different procedure is proposed which objective is to determine the Euclidean distance between the vectors of *csw*-efficiencies and *DEA super-efficiencies*.

Some definitions of *DEA* super-efficiencies can be found in the literature (see for

instance [40, 2, 26, 29]). Here, to compute the *DEA* super-efficiency (E_j) of a *DMU_j*, this *DMU_j* must be removed from the set of constraints (inequality (3.3) of the *DEACCR* Model and its efficiency score is maximized. Thus, in order to obtain a new target vector of *DEA* super-efficiencies, this procedure is repeated, one at a time, for each efficient *DMU* of Ψ . The remaining *DMUs*, out of Ψ , continue with their respective *DEA* efficiencies. Thus, note that a new target vector of *DEA* super-efficiencies allows that even efficient *DMUs* can improve their efficiency score and escape from the range area. Therefore, to restore efficiencies and relative sizes of all *DMUs*, Theorem 1 is used again to update their relative positions inside the range area.

In order to summarize this discussion, without loss of generality, a procedure for maximizing the *csw*-efficiencies of all *DMU's* of Ψ on the selfish approach is presented. The case of minimization of capital is analogous and it is omitted here. In order to accomplish that, first consider the following changes of *ModelMGE₂*, as detailed in the sequence:

$$\min \left(\sum_{j \in \Psi} \left(E_j^* - \frac{\sum_{r=1}^s \tilde{u}_r (Y_{rj} + \phi_{rj})}{\sum_{i=1}^m \tilde{v}_i A_{ij} (X_{ij} + \delta_{ij})} \right)^2 \right)^{1/2} \quad (5.18)$$

$$s.t. : \sum_{r=1}^s \tilde{u}_r (Y_{rj} + \phi_{rj}) - E_j^* \sum_{i=1}^m \tilde{v}_i A_{ij} (X_{ij} + \delta_{ij}) \leq 0; \quad \forall j \in \Psi. \quad (5.19)$$

$$\sum_{j \in \Psi} \left(\sum_{i=1}^m A_{ij} \delta_{ij} \right) = \bar{c}_0; \quad (5.20)$$

$$\left(\sum_{r=1}^s \tilde{u}_r (Y_{rj} + \phi_{rj}) \right) \left(\sum_{i=1}^m \tilde{v}_i A_{ij} (X_{ij} + \delta_{ij}) \right) \geq 2t_j \tilde{s}_j; \quad \forall j \in \Psi. \quad (5.21)$$

$$\delta_{ij} \geq -X_{ij} + L_{ij}; \quad \forall i = 1, \dots, m \text{ and } j \in \Psi. \quad (5.22)$$

$$\phi_{rj} \geq -Y_{rj} + U_{rj}; \quad \forall r = 1, \dots, s \text{ and } j \in \Psi. \quad (5.23)$$

This formulation, including super-efficiencies and on the selfish approach, is denoted by *ModelSE₂*. Similarly, it can be defined from *ModelMGE₁* a formulation of super-efficiency on the altruistic approach, denoted here by *ModelSE₁*.

Note in the formulation above that *DEA* super-efficiencies are dealt as a target instead of *DEA* efficiencies (objective function 5.18). Essentially, the idea is to update the Euclidean distance between the vector of *csw*-efficiencies and vector $E_j^* = (E_1^*, \dots, E_n^*)$ $\forall j \in \Psi$. In the constraint (5.19), we assume that the new *csw*-efficiency of *DMU_j* is

always less or equal than E_j^* . In the constraint (5.20), the exact budget \bar{c}_0 assigned to all *DMUs* of Ψ is defined. Here, in the constraint (5.21), the upper bounds on the relative sizes for *DMUs* of Ψ are not imposed (obviously other constraints can be considered by the manager). Note that constraints (5.19) and (5.21) allow to escape from the range area. Finally, constraints (5.22) and (5.23) are lower bounds imposed on the variables.

Therefore, we state the following procedure for maximizing *csw*-efficiencies over Ψ on the selfish approach.

Algorithm 2: { Maximization of *csw*-efficiencies over Ψ - selfish approach }

Inputs: Matrices X, Y, A ; capital \bar{c}_0 ; subset Ψ ; lower bounds L, U and parameters $t_j > 0$ for $j \in \Psi$.

Outputs: Matrices \bar{X}, \bar{Y} ; capital c_j destined to DMU_j , $\forall j \in \Psi$; *csw*-efficiencies $e' = (e'_1, \dots, e'_n)$ and relative sizes $s' = (s'_1, \dots, s'_n)$.

Begin.

01. Compute normalized weights (\tilde{u}, \tilde{v}) ;
02. Compute ℓ_2 -efficiencies \tilde{e}_j and relative sizes \tilde{s}_j , for $j = 1, \dots, n$;
03. **For** $j = 1, \dots, n$ **do**
 - If** $j \in \Psi$ **then**
 - $E_j^* \leftarrow$ Compute *DEA* super-efficiency of DMU_j ;
 - else**
 - $E_j \leftarrow e_j^*$;
 - end if**;
04. $(\delta, \phi) \leftarrow \text{SolveModelSE}_2(X, Y, A, L, U, \Psi, \bar{c}_0, E^*, t_j, \tilde{u}, \tilde{v})$;
05. $(\bar{X}, \bar{Y}) \leftarrow \text{UpdateXY}(\delta, \phi, X, Y)$;
06. $(a'_j, b'_j) \leftarrow \text{FindVIVO}(\tilde{u}, \tilde{v}, \bar{X}, \bar{Y}, A) \forall j \in \Psi$;
07. Compute: $e'_j = b'_j/a'_j$ and $s'_j = (a'_j b'_j)/2 \forall j \in \Psi$;
08. $\alpha \leftarrow \max\{e'_j, \forall j \in \Psi\}$;
09. $\beta \leftarrow \max\{s'_j, \forall j \in \Psi\}$;
10. **If** $((\alpha > 1) \text{ or } (\beta > 1))$ **then** {evaluate super-efficiency or oversize}
 - $a \leftarrow \max\{1, \alpha\}$;
 - $b \leftarrow \max\{1, \beta\}$;
 - $k_e \leftarrow 1/a$ and $k_s \leftarrow 1/b$;
 - For** $\forall j = 1, \dots, n$ **do** {update efficiencies and/or relative sizes}

```


$$e'_j \leftarrow k_e e'_j;$$


$$s'_j \leftarrow k_s s'_j;$$

end for;
end if;
11. Return  $(e', s', c_j, \bar{X}, \bar{Y});$ 
end.

```

In the Steps 1 and 2, the normalized common set of weights (\tilde{u}, \tilde{v}) , efficiency scores \tilde{e}_j and relative sizes \tilde{s}_j for $j = 1, \dots, n$ are computed (see also Steps 1 to 7 of Algorithm 1). In the Step 3, the *DEA* super-efficiencies of every *DMU*_{*j*} of Ψ are determined and set $E_j = e_j^*$, for every $j \notin \Psi$. In Step 4, *ModelSE*₂ is executed to determine the changes (δ, ϕ) associated to *DMUs* of Ψ . In the Step 5, matrices \bar{X} and \bar{Y} are updated, and in Step 6 (a'_j, b'_j) , new virtual inputs and outputs for all *DMU*_{*j*} with $\forall j \in \Psi$, are computed. Note that, $a'_j = \tilde{a}_j$ and $b'_j = \tilde{b}_j$ (equivalently, $e'_j = \tilde{e}_j$ and $s'_j = \tilde{s}_j$) for every $j \notin \Psi$. In other words, the super-efficiency allows that only coordinates associated to *DMUs* of Ψ are updated. These coordinates will be used in the determination of α and β , respectively, the biggest efficiency and relative area of all *DMUs* of Ψ (Steps 7, 8 and 9). In the Step 10, if some efficiency score or relative area is bigger than *one*, parameters k_e and k_s are used to update their *csw*-efficiencies and relative sizes to new coordinates (e'_j, s'_j) inside the range area. Finally, in Step 11, the *csw*-efficiencies and relative sizes of all *n DMUs* and budget c_j assigned to every $j \in \Psi$ are returned.

Similarly to Algorithm 1, note that matrices \bar{X} and \bar{Y} (along with cost matrix A) define a new problem with another set of *DEA* efficiencies. Therefore, a new set of ℓ_2 -efficiencies can be computed again, through *ModelQP*. Thus, an interesting question is to investigate the relation between the *csw*-efficiencies of *ModelSE*₂ and the optimal efficiencies of *ModelQP* associated to matrices \bar{X} , \bar{Y} and A .

Example 2: In order to illustrate the models of subsections 5.1, 5.2 and 5.3, respectively, return to the example of Table 3.1. For instance, assume that $\Psi = \{DMU_3, DMU_4\}$ and all models run under altruistic approach. Once again some parameters must be set, related to lower bounds for inputs and outputs ($L_{13} = 1, L_{23} = 2, U_{13} = 1$) and ($L_{14} = 2, L_{24} = 3, U_{14} = 2$). In addition to this, assume that in the constraints the *DMU*₄ must keep its relative size, at least equal to $\tilde{s}_4 = 0.80$ ($t_4 = 1$) while *DMU*₃ with $\tilde{s}_3 = 0.50$ is admitted to become 10% smaller ($t_3 = 0.9$). For short, the models are referred by their

last characters in Table 5.1 (for instance, we change *ModelMTC*₁ by *MTC*₁).

Models	c_0	α	β	<i>DMU</i> ₁		<i>DMU</i> ₂		<i>DMU</i> ₃			<i>DMU</i> ₄			<i>DMU</i> ₅	
				e'_1	s'_1	e'_2	s'_2	c_3	e'_3	s'_3	c_4	e'_4	s'_4	e'_5	s'_5
<i>MTC</i> ₁	-11.28	1.00	1.00	0.60	0.13	0.15	0.13	-2.30	1	0.45	-8.98	0.22	0.80	0.32	1.00
<i>MGE</i> ₁	-11.28 (\bar{c}_0)	1.00	1.00	0.60	0.13	0.15	0.13	-2.30	1	0.45	-8.98	0.23	0.82	0.32	1.00
<i>SE</i> ₁	-11.28 (\bar{c}_0)	3.12	1	0.19	0.13	0.05	0.13	-2.12	1	0.45	-9.16	0.07	0.80	0.10	1.00

Table 5.1: Management of the Total Capital with $\Psi = \{DMU_3, DMU_4\}$. *Altruistic approach* with 5 *DMU*'s

Firstly, the *ModelMTC*₁ (see subsection 5.1) is dealt. Assume in the constraints a new vector of *csw*-efficiencies 20% farther from its target solution ($e_j^*, \forall j = 1, \dots, n$), in other words, set $t_d = 1.2$. This means that overall or global efficiency (function of the Euclidean distance), decreases from 1.256^{-5} to 1.507^{-5} . This situation is possible whenever the coalition of *DMUs* of Ψ may face, for instance, to some adverse circumstances when they are prevented to perform as regularly. Therefore, the *ModelMTC*₁ searches for the lowest total budget required for all *DMUs* of Ψ to properly satisfy the constraints. In this example, since $t_d = 1.2$, is admitted that any *DMU* of Ψ could present even a worsening on its efficiency score to help the entire set of *DMUs*. The model found that \$11.28 monetary units could be saved, distributed on \$2.30 for *DMU*₃ and \$8.98 for *DMU*₄. The new relative sizes remain on their lowest bound, respectively, $s'_3 = 0.45$ and $s'_4 = 0.80$. The efficiency scores are kept ($e'_3 = \tilde{e}_3 = 1$, $e'_4 = \tilde{e}_4 = 0.22$) (see Table 5.1). A fact to be observed is that the budget saving distribution remains strongly on *DMU*₄ ($\bar{c}_4 = -8.98$), explained by its more expensive costs. The values of α and β (up to *one*) ensure that we have no displacements out of the range area.

Secondly, it is expected to maximize the global efficiency of all *DMUs* of Ψ (Subsection 5.2). The *ModelMGE*₁ is applied to an instance with the same budget found previously ($\bar{c}_0 = -\$11.28$), and kept all other constraints on the relative size. In this scenario, the model seems to work properly, with the same previous capital saving $\bar{c}_3 = -\$2.30$, efficiency score $e'_3 = 1$ and relative size $s'_3 = 0.45$ for *DMU*₃. For *DMU*₄, although keeping the same capital saving ($\bar{c}_4 = -\$8.98$), the efficiency score slightly increased to $e'_4 = 0.23$ and relative size to $s'_4 = 0.82$. In these first models, the new values found for efficiencies and relative sizes of all *DMUs* of Ψ are kept in the range area ($\alpha = 1.00, \beta = 1.00$). Hence, there is no variation on these attributes for remaining *DMUs* out of Ψ . In Table 5.1, the values in bold are associated with the objective function being considered.

Finally, in model *ModelSE*₁ super-efficiency is addressed. Here, it is allowed that *DMUs* of Ψ can escape from the range area. In present instance, after computing the *DEA* super-efficiencies of *DMU*₃ and *DMU*₄, respectively, are obtained $E_3^* = 3.12$ and $E_4^* = 0.23$. Note that *DMU*₃ extrapolates the range area (see Figure 5.1(a)). Therefore,

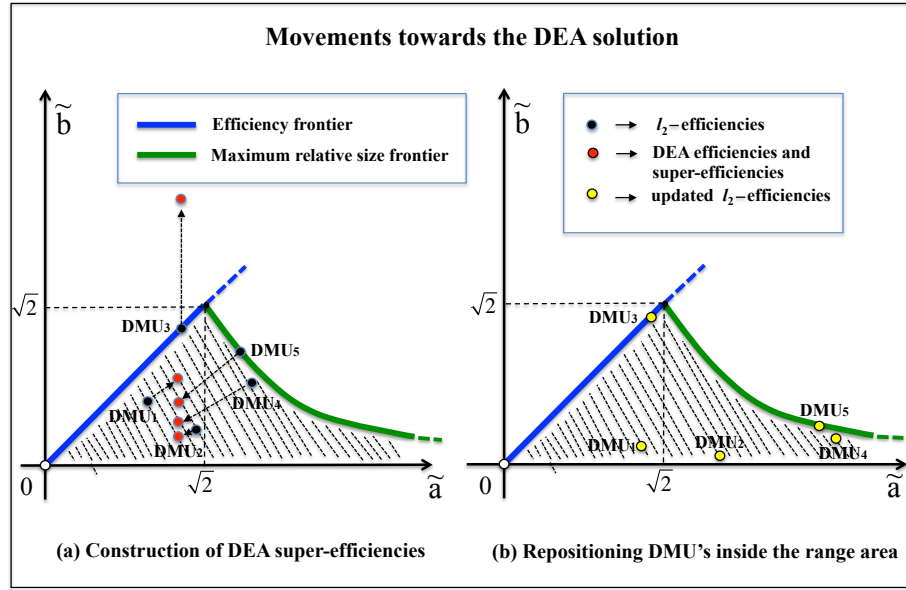


Figure 5.1: Movements towards the DEA solution. Subset $\Psi = \{DMU_3, DMU_4\}$. (a) Construction of DEA super-efficiencies (red points); (b) Repositioning all DMU 's inside the range area (yellow points).

$ModelSE_1$ indicates that DMU_3 still remains out of the range area ($\alpha = e'_3 = 3.12$) and Algorithm 2 can be executed in order to restore the coordinates of all DMU 's inside the range area (see Table 5.1 from columns 5 to 16). Thereafter, a displacement (down and to the right) of all DMU 's is observed. See the Figure 5.1(b). Observe that new relative sizes kept their lower bounds and the capital distribution is similar to $ModelMGE_1$, once again influenced by the cost matrix A . \square

5.4 Comments

Although the present models are nonlinear and non-convex constrained, they are regularly computed by some NLP solvers, which present the same results for each instance. There is not a model that deals with relative size in its objective function, due to a lack of theory that could develop this concept as well as that happened to efficiency, which would be a future direction to this research. Results obtained from MTC_1 and MGE_1 are acceptable and correspondent, which strengthens they use. It is evident that the vector of costs (A_{ij}) influence the capital sharing among DMU 's.

In the next chapter, an application with Brazilian health plans sector is developed.

Chapter 6

Brazilian Health Plans Modeling

Indeed, while hospitals/clinics are directly liable to health delivery, health plans are taking on a growing relevant role in countries health systems funding and, as a consequence, health care management itself. According to World Health Statistics 2015, between 2000 and 2012 private health plan global expenditure grew from 1.26% to 1.31% of Gross Domestic Product (GDP). In Brazil, at the same time, the GDP moved from 1.53% to 2.01%, which means, by 31.37%[43]. Since both providers and health plans are connected in the same economic chain, the latter performance must be meaningful and, thus, it would be reasonable to gauge also the health plans efficiency.

Brazilian supplementary health sector was chosen for an application of management of capital models mainly because it is well suited for non-parametric approach, when it is given the challenge in determination of its production function. Along with, it is a well known activity about which there are enough knowledge and expertise required in the outcome analysis. On the other hand, in Brazil, health plans sector became an important actor in country health funding, that in 2014 was liable for about R\$120 billions and 72 million people [3].

The management of capital models require three phases to be applied: first, DEA modeling must be performed with its related efficiencies identified; second, common set of weights, relative sizes and ℓ_2 -efficiencies must be computed; and, third, managerial scenario modeling must be set with outcome analysis. It is important to observe that in phases 1 and 2, original *DEA's CCR* input oriented model (chapter 2) and *KAO's* model (chapter 3) are extended with the introduction of costs parameters $A[i, j]$ ($\forall i = 1, \dots, m$ and $\forall j = 1, \dots, n$), in which m denotes the number of input variables and n , in turn, the number of *DMUs*.

6.1 Phase 1 - *DEA* Modeling

According to Golany&Roll[18], *DEA* modeling may be a three-step process: 1. Variables selection, when the appropriate inputs and outputs are identified; 2. *DMUs* selection, where the target is defined based on *DEA* principles and available data; and, 3. model choice, that concerns with the most suited *DEA* approach, according to the characteristics of the production system being modeled, and orientation, that determines whether is better to maximize outputs or minimize inputs.

6.1.1 Variables Selection

The process involving the determination of inputs and outputs is characterized by two procedures: first, a start list of variables is created; second, the suggested variables are assessed [18].

In order to obtain the variables start list, Almeida2010 [1] proposes a *DEA* literature review related to object area. Thus, a *DEA* applied to health plan literature survey was performed and also headed by a specific search proceeded in Scopus and Web of Knowledge index engines using the combined terms "Health+*DEA*" and "Health+Data Envelopment Analysis" with 394 results for former combination and 137 for latter. After a selective reading, 9 papers were admitted to be related to health plan and *DEA*, which can be seen in Table 6.1, exposing their number of citations, kind of efficiency addressed and country of the study.

Table 6.1: Health plan *DEA* literature survey

Reference	Cited	Efficiency Addressing	Country
Bryce1994[6]	40	Performance	USA
Rosenman1997[36]	19	Performance	USA
Siddharthan2000[39]	17	Performance	USA
Rollins2001[35]	7	Performance	USA
Brockett2004[5]	6	Performance	USA
Yang2006[44]	26	Performance	Canada
Sampaio2007[38]	0	Performance	Brazil
Fernandes2007[14]	3	Performance	Brazil
Carrington2011[7]	0	Performance	Australia

All papers are concerned with efficiency as operational performance and USA (55.5%) is the most frequently studied country, which concentrated all the attention in the beginning, since Bryce1994[6] with a *HMO's* comparative efficiency study using *DEA*, stochastic production frontier and fixed-effects regression, until Brockett2004[5], in which *DEA*

is used to assess the comparative efficiency between indemnity health insurance providers and *HMO*, through the provider's autonomy. The latest paper is from 2011, which assess the evolving of premiums paid in Australian's *PHI* (Private Health Insurance) system regarding the companies efficiency[7]. There are two papers about Brazilian private health plans[38][14] which assess the sector in terms of how efficiency is determined by managed care and from an economic/financial perspective.

Table 6.2: Health plans attributes survey

ATTRIBUTES		PAPER(S)	1 Bryce1994	2 Rosenman1997	3. Siddharthan2000	4 Rollins2001	5 Brockett2004	6 Yang2006	7 Fernandes2007	8 Sampaio2007	9 Carrington2011	FREQUENCY
METHODOLOGY CLASS	Comparative <i>DEA</i>		x									1
	Extended <i>DEA</i>						x					1
	Classic <i>DEA</i>								x	x		2
	Two-stage <i>DEA</i>			x	x	x		x			x	5
NON- <i>DEA</i> TECHNIQUE	Stochastic Production Frontier / Fixed Effects Regression		x									1
	Malmquist Index										x	1
	Theory of Games						x					1
	Regression Analysis				x	x						2
	Multivariate Analysis			x								1
	None							x	x	x		3
<i>DEA</i> MODEL	<i>CCR</i>			x	x			x			x	4
	<i>BCC</i>		x					x	x	x	x	5
ORIENTATION	Input			x	x			x	x		x	5
	Output							x	x			2
EFFICIENCY	Performance		x	x	x		x	x	x	x	x	8
	Quality											0
SET SIZE	< 50			x		x					x	3
	≥50, <100							x				1
	≥100, <200						x					2
	≥200, <500		x		x			x				2
	≥500, <1000											0
	>1000							x	x			2
DATA SOURCE	Primary											0
	Secondary		x	x	x	x	x	x	x	x	x	8

This literature survey encompasses how *DEA* is used to model health plans regarding methodology class, non-*DEA* associated technique, *DEA* model and orientation, efficiency approach, set size, data source and input/output variables. Some input and output variables had to be adjusted to a common denomination or meaning, so they could be compiled together. As can be seen in Table 6.2, a two-stage *DEA* is the main methodology (55.5%) carried with several different non-*DEA* technologies. The second stage often uses a technique to examine explanatory variables influence over the observations set, as multivariate analysis to determine the influence of models types and ownership over Florida's *HMO's*[36] or Malmquist Index to introduce efficiency measuring in raising premiums analysis over time in Australian private health sector[7]. *BCC* model, to isolate the pure technical efficiency, is more used (55.5%) and input orientation is predominant, which highlights the focus in minimizing inputs rather to maximize outputs, which agrees with the larger input variable set identified.

There are 18 different input variables, emphasizing administrative and medical costs (44.4%, each one), which evidences the minimizing costs concern (Table 6.3). Capital

Table 6.3: Input health plans variables survey

PAPER(s)	INPUT VARIABLES	Impatient days	Maternity days	Newborn days	Outpatient visits	Emergency room visits	Procedures performed	Number of physicians	Administrative costs	Assistencial costs	Total costs	Cash/Banks	Capital assets	Incomes	Indebtness	Reserve level	Tangible assets	Labour Expenses	Investments
1 Bryce1994		x			x			x	x										
2 Rosenman1997									x	x			x						
3. Siddharthan2000		x	x	x	x	x	x												
4 Rollins2001									x	x									
5 Brockett2004											x			x					
6 Yang2006										x	x		x			x		x	x
7 Fernandes2007												x	x	x	x	x	x		
8 Sampaio2007												x	x						
9 Carrington2011									x	x							x		
FREQUENCY		2	1	1	2	1	1	1	4	4	2	2	4	2	1	2	2	1	1

assets also appears with 44.4%, mainly because of Brazilian papers [38][14] which are focused on economic/financial issues. All other input variables are sparse over the input set. We can notice that some papers use typical assistance variables, as impatient days[6] and emergency room visits[39], putting health plans efficiency directly connected to health delivery levels.

In turn, output set has 7 variables and it is highly concentrated on the number of clients (66.6%) (Table 6.4), which shows how it composes the main proxy to health plan size. Indeed, number of clients (or *lives* as it is commonly said) is the primary focus of managerial decision in health plans area since it supports the break-even point overcome. The other data are sparse.

According to this literature review, thus, administrative and medical costs may be eligible as input variables, while, as output variable, the number of clients is prevalent. However, an important issue raised from the health plan sector refers to the optimization of premiums payment. As a business, a health plan must look for maximize this variable, but preserving the customers attainability. Trying to model this aspect, Income/Premiums variable will also be consider as a second output. A fact to be noted is the absence of any variable regarding quality issues, pointed by MoreyFL1990 [31] as a way to make broader comparative analysis. Keeping the assessment focus on performance undervalues what is actually perceived by the clients and society in medical services delivered by health plans.

Table 6.4: Output health plans variables survey

PAPER(s)	OUTPUT VARIABLES	Clients	Inpatient days	Outpatient visits	Working capital	Expenditures	Incomes\Premiums	Investment returns
1 Bryce1994		x						
2 Rosenman1997		x						
3. Siddharthan2000		x						
4 Rollins2001			x	x				
5 Brockett2004		x	x	x				
6 Yang2006							x	x
7 Fernandes2007					x	x		
8 Sampaio2007		x					x	
9 Carrington2011		x						x
FREQUENCY		6	2	2	1	1	2	2

Since variables start list is set, a method must be applied in order to choose the most appropriated variables among the suggested ones. These methods were developed aiming more reliable results and they are based on the identification of the most important and contributory variables of the object of study [1]. According to Dyson2001 [13], a lack of important variables is the first pitfall in variables selection. Based on variables homogeneity, a simple method is correlation analysis which enables to delete variables that are highly correlated, since they are redundant [21]. In order to mitigate subjectivity mistakes in variables exclusion some research propose a three stage analysis [42] [20]: 1. Compute all efficiency rates for selected variables; 2. Identify the meaningfulness level of each variable, withdrawing them slowly; and, 3. Compute efficiency rate of new data set. In opposite direction, called *stepwise*, Wagner&Shimshak2007 [41] proposes to start from a initial pair {input,output} and, little by little, adding new variables and computing the corresponding efficiency rate. For the purpose of this application, which mainly aims the applicability of management of capital models, the former method, correlation analysis is employed among administrative costs, medical costs, clients and incomes/premiums.

6.1.2 *DMUs* Sample

Since the variables are preliminarily identified, *DMUs* must be set. Because of high sensitiveness to outliers observations in *DEA*, *DMU's* set size must follow a tight relation

to the number of variables. Considering n , m and r to represent respectively, number of *DMUs*, inputs and outputs, there are two main approaches relating them, as follows:

1. Dyson2001 [13] proposes that $n \geq 2(m \times s)$; and,
2. Cooper&Seiford&Tone2000 [11] presents, in turn, that $n \geq \max\{m \times s; 3(m + s)\}$.

This way, latter expression is chosen in this application, thus *DMU's* set must be larger than 12. In Brazil, health plans data source is provided by ANS (National Agency of Supplementary Health) and it can be retrieved from *www.ans.gov.br*. At the end of 2014 there were 1.040 medical health plans active, classified into small (*clients* ≤ 20.000), medium (*clients* > 20.000 and *clients* ≤ 100.000) and large (*clients* > 100.000). For each group, 10 *DMUs* (Health Plans) are randomly selected as the base sample in this application.

6.1.3 *DEA* Model

According to chapter 3, capital management models require the use of *CCR* input oriented model.

6.1.4 Sample Assessment

Since the variables and *DMU's* characteristics are defined, the associated data must be retrieved and selected from *www.ans.gov.br*, referring to year of 2014. The input variables are the medical costs and the administrative costs while the number of clients and the income/premiums are the output variables. Medical and administrative costs mean the total sum of this expenses (in million of *reais*) over the year, number of clients refers to this amount at the end of the year (in thousand) and income/premiums refer to total costumers payments (in million of *reais*). One could argue about the expected strong correlation among inputs and among outputs, but the experience points that it is a admissible choice and, to mitigate it and provide a sharper and refined data comparison, a larger *DMU's* set is provided, with 30 health plans. In this application, as input variables are already based on monetary values, all $A[i, j]$ parameters are assumed to be 1.

The result data sample is presented in Table 6.5.

The efficiency scores obtained from *CCR* input model on the data sample are presented in Tables 6.6 and 6.7, while the related discriminant analysis in Table 6.8. *DMUs* A, E, H, W and C1 are considered *DEA* efficient and none of them is classified as small. On the other hand, *DMU* N presents the lowest efficiency (60%) and it is classified as small (12.000 lives) . That was expected to happen because small health plans present

Table 6.5: Health Plans Data Sample

DMU	Medical Costs	Administrative Costs	Clients	Income/Premiums	Size
A	11831	669	4084	13660	large
B	7116	562	1694	8610	large
C	468	30	218	525	large
D	243	29	154	333	large
E	196	30	324	261	large
F	221	27	62	233	medium
G	22	7	28	29	medium
H	44	2	41	47	medium
I	125	11	54	137	medium
J	88	7	45	97	medium
K	9	2	3	10	small
L	13	13	6	20	small
M	23	3	19	27	small
N	8	11	12	11	small
O	32	7	16	39	small
P	2402	193	1612	2588	large
Q	122	18	107	145	large
R	279	37	224	310	large
S	146	33	142	218	large
T	1150	146	569	1336	large
U	14	2	20	17	small
V	29	12	22	48	medium
W	7	4	32	16	medium
X	28	4	21	35	medium
Y	56	6	42	70	medium
Z	139	23	29	155	medium
A1	16	4	12	22	small
B1	16	3	16	21	small
C1	21	4	20	36	small
D1	14	7	10	20	small

huge challenges to operate due to its size, which prevent them to scale back office costs. Efficiency average is high (83%), the variance is 0.014 and standard deviation is 0.12, which it is compatible with society general appreciation of this sector, that makes no distinction among health plans services delivery.

Table 6.6: Efficiency Scores - *DMU's* A to O

DMU	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Efficiency	1.00	0.97	0.95	0.97	1.00	0.74	0.69	1.00	0.85	0.88	0.63	0.67	0.82	0.60	0.69

Table 6.7: Efficiency Scores - *DMU's* P to D1

DMU	P	Q	R	S	T	U	V	W	X	Y	Z	A1	B1	C1	D1
Efficiency	0.87	0.80	0.77	0.84	0.80	0.92	0.80	1.00	0.83	0.92	0.69	0.76	0.81	1.00	0.65

Table 6.8: Discriminant Analysis

Average	Standard Deviation	Covariance	Minimum
0.83	0.12	0.014	0.60

Table 6.9 presents the variables correlation in data sample. One can note that administrative and medical costs, as inputs, are strongly correlated (0.98) as well as clients and income/premiums, as outputs (0.96). Although literature review identified them as the most used in health plans sector, correlation results would lead to a choice among them, in order to reduce the variables set. However, experience says, for instance, that despite medical and administrative costs are correlated, the latter is much more influenced by managerial decisions than the former, which would suggest an efficiency aspect. At the time, even with clients and income/premiums in a tight correlation, one issue can arise with determinant relevance, that is, the ratio between income/premiums to clients, which point out how health plan are efficient in charging their costumers. In order to mitigate correlation issues, by making these ones not obvious but crucial efficiency aspects to be assessed, a larger *DMU* set is built so *DEA* technique can be sharper in determining relative efficiencies. Indeed, in this sample, while ratio administrative costs to medical costs is, in average, about 5% in efficient *DMUs*, it is by 137% in worst efficient one, and it is a very important managerial observation in this kind of business. On the other hand, the ratio income/premiums to clients reaches 3.34 in efficient *DMUs* while, in the worst one, it is only about 0.91. These findings are absolutely relevant in health plan business and support the present variables selection.

Table 6.9: Variables Correlation

Variables	Medical Costs	Administrative Costs	Clients	Income/Premiums
Medical Costs	1	-	-	-
Administrative Costs	0.98	1	-	-
Clients	0.96	0.94	1	-
Income/Premiums	0.99	0.98	0.96	1

6.2 Phase 2 - Relative sizes and ℓ_2 -efficiencies

Next step consists on identifying relative sizes and ℓ_2 -efficiencies, which are central concepts in further management of capital modeling. Aiming this, a common set of weights must be computed before, following what was presented in chapter 3 in which an extended $KAO's$ model is used. Thus, $\bar{v}_1 = 0.00059261$ and $\bar{v}_2 = 0.00187859$ were found as medical and administrative costs weights, respectively, denoting that the latter is more considered by the model than the former. Besides, clients and income/premiums are weighted with $\bar{u}_1 = 0.0001$ and $\bar{u}_2 = 0.000498865$, showing an inclination to the last one. As results, just DMU C1 ℓ_2 -efficiency reaches its DEA efficiency ($\bar{e} = e^* = 1$). Some other $DMU's$ ℓ_2 -efficiencies are very far from their correspondent DEA efficiencies as, for example, $DMU's$ L and N, both of them are classified as small.

Regarding relative area issue, one DMU (A) appears with the most expressive value (29.85), almost three times larger than the second one. Indeed, DMU A is one of the largest health plans in Brazil and it is the *maximum relative size* ($\tilde{s}_A = 1$). This discrepancy makes other $DMUs$ to present very small amounts of relatives sizes, which brings some difficulties in dealing with this parameter in further assessment.

Finally, the common set of weights need to be normalized in order to be used in further capital management models. In order to accomplish that, the ratio of vectors (\bar{v}_i, \bar{u}_r) to $\sqrt{\bar{r}} = 29.85$, in which \bar{r} is the largest relative area, must be computed. At the end, are obtained $\tilde{v}_1 = 0.00010845$, $\tilde{v}_2 = 0.000343789$ and $\tilde{u}_1 = 0.0000183$, $\tilde{u}_2 = 0.0000912943$ as input and output variables normalized weights, respectively. Now, haven ℓ_2 -efficiencies, relatives sizes and normalized common set of weights already identified, capital management models prerequisites are filled.

6.3 Phase 3 - Management of Capital

Management of capital is based on relationship among relative efficiency, relative size and a budget availability mediated by mathematical programming models. However, in

Table 6.10: Relative sizes and ℓ_2 -efficiencies

DMU	<i>DEA</i> Efficiency	ℓ_2 -efficiency	Relative Area	Relative Size
A	1.00	0.87	29.85	1.00
B	0.97	0.85	11.77	3.94×10^{-1}
C	0.95	0.85	0.0473	1.58×10^{-3}
D	0.97	0.91	0.0180	6.03×10^{-4}
E	1.00	0.94	0.0140	4.70×10^{-4}
F	0.74	0.67	0.0111	3.72×10^{-4}
G	0.69	0.66	0.0002	7.58×10^{-6}
H	1.00	0.92	0.0004	1.37×10^{-5}
I	0.85	0.78	0.0035	1.17×10^{-4}
J	0.88	0.81	0.0017	5.78×10^{-5}
K	0.63	0.58	0.00002	8.05×10^{-7}
L	0.67	0.33	0.00002	5.69×10^{-6}
M	0.82	0.80	0.00001	4.96×10^{-6}
N	0.60	0.26	0.00001	2.85×10^{-6}
O	0.69	0.66	0.00003	1.13×10^{-5}
P	0.87	0.81	1.29	1.74×10^{-5}
Q	0.80	0.78	0.0044	1.47×10^{-4}
R	0.77	0.75	0.0208	6.96×10^{-4}
S	0.84	0.83	0.0091	3.06×10^{-4}
T	0.80	0.76	0.3457	1.15×10^{-2}
U	0.92	0.87	0.0001	2.11×10^{-6}
V	0.80	0.66	0.0005	1.74×10^{-5}
W	1.00	0.96	0.0001	2.19×10^{-6}
X	0.83	0.81	0.0002	7.86×10^{-6}
Y	0.92	0.88	0.0009	2.91×10^{-5}
Z	0.69	0.63	0.0050	1.68×10^{-4}
A1	0.76	0.72	0.0001	3.46×10^{-6}
B1	0.81	0.80	0.0001	3.06×10^{-6}
C1	1.00	1.00	0.0002	6.67×10^{-6}
D1	0.65	0.51	0.0001	3.94×10^{-6}

order to make it work, some parameters must be set, related to input costs and inputs and outputs variations bounds, admitted in managerial scenario. In this applicatin, as input variables are already expressed with monetary values, all costs are assumed to be 1 ($A[i, j] = 1, \forall i \in 0..m, \forall j \in 0..n$). On the other hand, all new inputs and outputs (\bar{X}, \bar{Y}) are admitted to be, at least, the half of original values ($\bar{X} \geq \frac{X}{2}, \bar{Y} \geq \frac{Y}{2}$). Finally, 6 *DMUs* are randomly chosen from the *DMU* sample as the object of further management of capital applications (Table 6.11), each 2 from a health plans clients size category.

Table 6.11: *DMUs* subsample for management of capital application

Size	Medical Costs	Administrative Costs	Clients	Income/Premiums	DEA Efficiency	ℓ_2 -efficiency	Relative Size	Category
A	11831	669	4084	13660	1.00	0.87	1.00	large
C	468	30	218	525	0.95	0.85	1.58×10^{-3}	large
F	221	27	62	233	0.74	0.67	3.72×10^{-4}	medium
C1	21	4	20	36	1.00	1.00	6.67×10^{-6}	medium
K	9	2	3	10	0.63	0.58	8.05×10^{-7}	small
N	8	11	12	11	0.60	0.26	2.85×10^{-6}	small

This application is organized as follows in next sections: first, all models for a single *DMU* are applied to each one of Table 6.11; second, the models for subset of *DMUs* are applied. For each application, the same parameters rates are used and *DMU's* behaviors are inspected and compared, observing specific aspects of health plans business. A *DEACM* (*Data Envelopment Analysis Capital Management*) software, to perform management of capital modeling, was developed. Now it is hosted in *AWS* (*Amazon Web Services*) cloud service and can be accessed in <http://52.91.19.81:8080/deacm>. The following technologies were employed in its development:

- *Java 8*: programming language;
- *JHipster*: Java projects generator, with Spring Boot and AngularJS;
- *Spring Boot*: Java framework for corporate applications development based on convention rather than configuration;
- *AngularJS*: Javascript framework for 'web single-page' applications development;
- *AMPL*: a language for mathematical programming language;
- *Apache Commons Math*: a mathematical/statistics components library;
- *Apache TomCat*: a web server;
- *Apache Commons CSV*: a library for CSV files processing;
- *Apache Commons IO*: a library for I/O tasks;

- *Bower*: a web packages manager;
- *Maven*: Java projects dependencies manager;
- *BootStrap*: CSS framework for web applications;
- *BootStrap Material Design*: bootstrap theme for Googles's "material design" style.

6.3.1 Single *DMU* - Minimization of Capital

As we already know, management of capital models deal with the relations among capital (budget), relative size and *csw*-efficiency. In this subsection, given relative size and *csw*-efficiency parameters as goals, *ModelMC* searches to the minimum capital to achieve them. Here, in order to make possible the results analysis and comparison, four combinations of parameters are applied for each *DMU* in focus, with relative size rate assuming values 1.1 or 0.9 ($K_s = 1.1, K_s = 0.9$) and *csw*-efficiency rates also presenting the same values ($K_e = 1.1, K_e = 0.9$), covering all possible movements in the range area.

Table 6.12: Minimization of Capital - $DMU_0 \equiv A$.

Capital	K_s	K_e	Size	<i>csw</i>	Range Area	Inputs Change		Outputs Change	
						Medical Costs	Administrative Costs	Clients	Income/Premiums
-4044	1.1	1.1	1	0.96	II	-5916	1872	0	1415
-3579	1.1	0.9	1	0.78	I	-5916	2336	0	-102
-4464	0.9	1.1	1	0.96	IV	-5916	1451	0	-102
-4044	0.9	0.9	1	0.79	III	-5916	1872	0	-1475

For $DMU_0 \equiv A$, one can observe that all instances bring a negative capital (budget), meaning that, despite the movement in the range area, a saving can always be done (Table 6.12). *DMU A* is the *maximum reference size* ($\tilde{s}_A = 1$) and, indeed, it is one of the biggest Brazilian health plans. Because of this, in the two first instances in which ($K_s > 1$) a relative oversize happens and, then, Algorithm 1 (chapter 4, section 4.3) must be applied. The displacements in the range area are previously defined, along with K_s and K_e variations. It is clear that the model aims to minimize Medical Costs input because it presents larger amounts when minimization effort is more efficient. On the other hand, Administrative Costs always increases, which is not so desirable to business but it is necessary to accomplish mainly relative size goals. It is interesting to note that the same capital is obtained as well for $K_s = 1.1$ and $K_e = 1.1$ (increasing) as for $K_s = 0.9$ and $K_e = 0.9$ (decreasing) and it is a consistent outcome.

In the sequence, analyzing another large health plan ($DMU_0 \equiv C$) it is clear that it follows the same behavior of the previous one (Table 6.13). Once again the minimization effort falls on Medical Costs, because of its larger amounts, which agrees with a reasonable

Table 6.13: Minimization of Capital - $DMU_0 \equiv C$.

Capital	K_s	K_e	Size	csw	Range Area	Inputs Change		Outputs Change	
						Medical Costs	Administrative Costs	Clients	Income/Premiums
-160	1.1	1.1	1.74×10^{-3}	0.94	II	-234	74	0	56
-141	1.1	0.9	1.74×10^{-3}	0.77	I	-234	92	0	-3
-177	0.9	1.1	1.42×10^{-3}	0.94	IV	-234	57	0	-3
-160	0.9	0.9	1.42×10^{-3}	0.77	III	-234	74	0	-56

managerial decision. All instances present some increase in Administrative Costs input, which is needed in order to reach relative size goals. As this input has a strong weight, it makes sense to use it in health plan size balancing due to a less use of capital. Since Income/Premiums is more important than Clients, the model prefers the former when dealing with output variables. Indeed, in Brazilian health plans market, strongly based on corporate contracts, it seems that is better rather to improve the ratio income/premiums to clients than to look for new clients.

Table 6.14: Minimization of Capital - $DMU_0 \equiv F$.

Capital	K_s	K_e	Size	csw	Range Area	Inputs Change		Outputs Change	
						Medical Costs	Administrative Costs	Clients	Income/Premiums
-76	1.1	1.1	4.09×10^{-4}	0.74	II	-111	35	0	23
-65	1.1	0.9	4.09×10^{-4}	0.61	I	-111	45	0	-2
-85	0.9	1.1	3.35×10^{-4}	0.74	IV	-111	26	0	-2
-76	0.9	0.9	3.35×10^{-4}	0.61	III	-111	35	0	-25

Next two health plans ($DMU_0 \equiv F$ and $DMU_0 \equiv C1$), though classified as mediums, present quite different behaviors. While DMU F throughly accompanies the first two large ones, mainly because it still has many clients (Table 6.14), DMU C1 is in the border between medium and small health plans. Thus, in two similar instances in which both parameters relative size and csw -efficiency are increasing ($K_s = 1.1, K_e = 1.1$) or decreasing ($K_s = 0.9, K_e = 0.9$), *ModelMC* indicates the use of Clients output variable.

Table 6.15: Minimization of Capital - $DMU_0 \equiv C1$.

Capital	K_s	K_e	Size	csw	Range Area	Inputs Change		Outputs Change	
						Medical Costs	Administrative Costs	Clients	Income/Premiums
-7.5	1.1	1.1	7.30×10^{-6}	1	II	-11	3.5	-3.6	4.7
-6.4	1.1	0.9	7.30×10^{-6}	0.94	I	-11	4.5	0	-0.2
-8.5	0.9	1.1	6.00×10^{-6}	1	IV	-11	2.4	0	-0.2
-7.5	0.9	0.9	6.00×10^{-6}	0.94	III	-11	3.5	-4	-3.2

DMU C1 is csw -efficient ($e' = 1$) and there is a superefficiency occurrence in the first instance with the respective use of Algorithm I (chapter 4, section 4.3) to put the DMU back in the range area. This must be the factor which induces the use of Clients, when extreme situations are placed and a health plan already efficient has to use up the least important weight to reach the goals (Table 6.15).

The last two health plans are the smaller ($DMU_0 \equiv K, DMU_0 \equiv N$). Despite K be the smallest DMU , it still follows the standard behavior of other $DMUs$ (except DMU

C1). This happens because of its *csw*-efficiency ($e' = 0.58$) is not far from others (Table 6.16).

Table 6.16: Minimization of Capital - $DMU_0 \equiv K$.

Capital	K_s	K_e	Size	<i>csw</i>	Range Area	Inputs Change		Outputs Change	
						Medical Costs	Administrative Costs	Clients	Income/Premiums
-2	1.1	1.1	9.00×10^{-7}	0.64	II	-3	1	0	1
-1.6	1.1	0.9	9.00×10^{-7}	0.52	I	-3	1.5	0	0
-2.4	0.9	1.1	7.00×10^{-7}	0.64	IV	-3	0.5	0	0
-2	0.9	0.9	7.00×10^{-7}	0.52	III	-3	1	0	-1

On the other hand, *DMU N* presents the least *csw*-efficiency ($e' = 0.26$) and it is not possible to *ModelMC* to reach a size increase of 10% limited to a *csw*-efficiency increase of, at most, also 10%. It would require a larger efficiency rate and, because of this, the two first instances are infeasible. Indeed, this health plan presents a catastrophic situation in which the administrative costs overcome the medical costs. It could be said that it is driven to bankruptcy. In the last instances *DMU N* still behaves in its own way, decreasing Administrative Costs in order to decrease relative size and only decreasing Income/Premiums in order to decrease *csw*-efficiency (Table 6.17).

Table 6.17: Minimization of Capital - $DMU_0 \equiv N$.

Capital	K_s	K_e	Size	<i>csw</i>	Range Area	Inputs Change		Outputs Change	
						Medical Costs	Administrative Costs	Clients	Income/Premiums
-	1.1	1.1	-	-	-	-	-	-	-
-	1.1	0.9	-	-	-	-	-	-	-
-1.2	0.9	1.1	2.60×10^{-6}	0.29	IV	0	-1.2	0	-0.14
0.26	0.9	0.9	2.60×10^{-6}	0.24	III	0.26	0	0	-1.4

Finally, it can be said the the Minimization of Capital model works properly, trying, in most of situations, to save capital through the input with larger amounts (Medical Costs) and keeping the variable with least important weight to extreme situations (Clients), mainly when *DMU* is already *csw*-efficient, which is suitable to health plans managerial decisions. Along with, Administrative Costs input variables, with strong weight, is employed moderately to balance the capital minimization process.

6.3.2 Single *DMU* - Maximization of Size

The start decision in order to experiment the Maximization of Size model (*ModelMS*) is to use the results obtained in subsection 6.3.1 as capital parameter (\bar{c}). Thus, it can be observed if this model presents results compatible to those ones from *ModelMC*. In this direction *csw*-efficiency parameter set ($K_e = 1.1, K_e = 0.9$) is repeated twice for each *DMU*.

As well as in the minimization of capital of $DMU_0 \equiv A$, there is oversize situation in two first instances, which is properly expected (Table 6.18). Besides Administrative Costs in second instance and Income/Premiums in first instance, there are not other differences in any values, which strengthens the correspondence between *ModelMC* and *ModelMS*.

Table 6.18: Maximization of Size - $DMU_0 \equiv A$.

Size	K_e	Capital	csw	Range Area	Inputs Change		Outputs Change	
					Medical Costs	Administrative Costs	Clients	Income/Premiums
1	1.1	-4044	0.96	II	-5916	1872	0	1416
1	0.9	-3579	0.79	I	-5916	2337	0	-101
1	1.1	-4464	0.96	II	-5916	1452	0	-101
1	0.9	-4044	0.79	I	-5916	1872	0	-1475

In turn, for $DMU_0 \equiv C$ although values are very close, some slight differences can be found (Table 6.19). First, the relative size differs in 0.00000590, 0.00001030, 0.00000410 and 0.00000410 units respectively from first to fourth instances, that means a 0.5% variation, absolutely acceptable. Besides, the first three results for Income/Premiums are bigger than from *ModelMC*. Once again, they are not so meaningful since the variation is about 5%, which does not commit the results (Table 6.19).

Table 6.19: Maximization of Size - $DMU_0 \equiv C$.

Size	K_e	Capital	csw	Range Area	Inputs Change		Outputs Change	
					Medical Costs	Administrative Costs	Clients	Income/Premiums
1.75×10^{-3}	1.1	-160	0.94	II	-234	74	0	59
1.75×10^{-3}	0.9	-141	0.77	I	-234	93	0	-1.3
1.43×10^{-3}	1.1	-177	0.94	IV	-234	57	0	-2.1
1.43×10^{-3}	0.9	-160	0.77	III	-234	74	0	-56

The next two DMU' s are medium health plans, thus some change is expected regarding the first two, which are large health plans. Along with what happened in *ModelMC*, DMU F completely follows the behavior of DMU' s A and C. Indeed, a similar change in relative size occurred, that is, -0.00000170 , 0.00000440 , 0.00000030 and -0.00000140 respectively for each instance, which remains irrelevant (1%). Besides, there is only one difference in second instance for Income/Premiums. All other results match exactly to *ModelMC* (Table 6.20).

Table 6.20: Maximization of Size - $DMU_0 \equiv F$.

Size	K_e	Capital	csw	Range Area	Inputs Change		Outputs Change	
					Medical Costs	Administrative Costs	Clients	Income/Premiums
4.08×10^{-4}	1.1	-76	0.74	II	-111	35	0	23
4.14×10^{-4}	0.9	-65	0.61	I	-111	46	0	-0.6
3.35×10^{-4}	1.1	-85	0.74	IV	-111	26	0	-2
3.33×10^{-4}	0.9	-76	0.61	III	-111	35	0	-25

As already registered, DMU C1 is csw -efficient ($e' = 1$) thus some special characteristics are expected to happen. First, the difference in relative sizes are larger, mainly in first instance (20%), when superefficiency occurred and *ModelMS* decided not to exhaust

Medical Costs lower bound and, on the other hand, not to increase Administrative Costs too much. It may be an intelligent decision on scenario without growth in Clients and Income/Premiums. It must be remembered that *DMU* C1 is in the frontier between medium and small health plans (Table 6.21).

Table 6.21: Maximization of Size - $DMU_0 \equiv C1$.

Size	K_e	Capital	csw	Range Area	Inputs Change		Outputs Change	
					Medical Costs	Administrative Costs	Clients	Income/Premiums
6.10×10^{-6}	1.1	-7.5	1	II	-9.5	2	0	0
7.40×10^{-6}	0.9	-6.4	0.94	I	-11	4.6	0	-0.14
6.10×10^{-6}	1.1	-8.5	1	IV	-11	2.5	0	0
6.10×10^{-6}	0.9	-7.5	0.94	III	-11	3.5	0	-3.8

At last, small health plans are experimented. *DMU* K reaches the previous relative size in instance three and four. In first two it did not happen. It must be remembered that *DMU* K is the smallest one and it must be very difficult to grow, specially in a situation of capital saving. A factWhat can be observed is that the model does not exhaust Medical Costs, using more Administrative Costs to handle the goal achievement. Also, it does not deal with output variables change (Table 6.22).

Table 6.22: Maximization of Size - $DMU_0 \equiv K$.

Size	K_e	Capital	csw	Range Area	Inputs Change		Outputs Change	
					Medical Costs	Administrative Costs	Clients	Income/Premiums
7.00×10^{-7}	1.1	-2	0.64	II	-3	0.3	0	1
7.00×10^{-7}	0.9	-1.6	0.52	I	-3	1.5	0	0
7.00×10^{-7}	1.1	-2.4	0.64	IV	-3	0.5	0	0
7.00×10^{-7}	0.9	-2	0.52	III	-3.7	1.7	0	0

Since *DMU* N presents a special situation, in which Administrative Costs overcome Medical Costs, it is expected that their results are also different. Indeed, no movement happens with Administrative Costs which is along with a managerial decision in a unbalanced situation. One other aspect to be considered is that relative size seems to be directly influenced by *csw*-efficiency. The relative sizes match exactly the values of *ModelMC*, and there are few differences in variables amount in third instance (Table 6.23).

Table 6.23: Maximization of Size - $DMU_0 \equiv N$.

Size	K_e	Capital	csw	Range Area	Inputs Change		Outputs Change	
					Medical Cost	Administrative Costs	Clients	Income/Premiums
2.90×10^{-6}	1.1	-1.2	0.29	II	-1.2	0	0	0.75
2.60×10^{-6}	0.9	0.26	0.24	I	0.26	0	0	-1.4
2.90×10^{-6}	1.1	-1.2	0.29	IV	-1.2	0	0	0.75
2.60×10^{-6}	0.9	0.26	0.24	III	0.26	0	0	-1.4

Concluding the experiment, *ModelMS* seems to work with a strong correspondence with *ModelMC*, which strengthens both validation and movements in the range area remain the same. It was not expected and it is nor even interesting that models inserted in

the same context present discrepant or antagonistic results. In specific and border situations or due to the objective function some differences are observed but totally explained, restricted to *csw-efficient DMU's*.

6.3.3 Single *DMU* - Maximization of Efficiency

In the same direction of subsection 6.3.2, the capital obtained in *ModelMC* experiments are used as parameters values in maximization problems. The other parameter is relative size rate whose values are $K_s = 1.1$ and $K_s = 0.9$ and they are applied twice each one. The *csw-efficiency*, however, are not expected to match, in general, the values dealt in previous models. That happens because *ModelME* is able to continue manipulating capital distribution among inputs variables, even when parameters relative size and capital are already reached, searching for better efficiencies results. That is, efficiency limits much more relative size and capital than the opposite. In order to match the same previous results, an efficiency minimization model is required, which brings the minimum efficiency rate to accomplish the goals. In present situation, in general, the displacements to areas I and III are not expected, which would mean efficiency decrease.

Observing $DMU_0 \equiv A$, one can note that *ModelME* aims to exhaust Administrative Costs (the highest *KAO's* normalized weight) before using the Medical Costs, which is along with managerial decision to decrease any costs that are not directly connected to target (health care delivering), in order to make the business more efficient. Along with, *ModelME* also indicates the increase of Income/Premiums, which is an expected decision too (Table 6.24). All instances present superefficiency while the first two also present relative oversize, when Algorithm I is applied in order to put *DMU A* back in range area.

Table 6.24: Maximization of Efficiency - $DMU_0 \equiv A$.

<i>csw</i>	K_s	Capital	Size	Range Area	Inputs Change		Outputs Change	
					Medical Costs	Administrative Costs	Clients	Income/Premiums
1	1.1	-4044	1	II	-3709	-335	0	9731
1	1.1	-3579	1	II	-3244	-335	0	8563
1	0.9	-4464	1	IV	-4129	-335	0	6277
1	0.9	-4044	1	IV	-3709	-335	0	5328

As can be seen with *DMU A*, the experiment with $DMU_0 \equiv C$ leads to similar results. Once again, *ModelME* uses all Administrative Costs available, that is, goes to the lower bound. There is superefficiency in all instances. It must be registered that this behavior is absolutely expected, since *DMU's A* and *C* are in the same category, and may present similar operational characteristics (Table 6.25).

The next two medium health plans F and C1, repeat the same behavior observed

Table 6.25: Maximization of Efficiency - $DMU_0 \equiv C$.

csw	K_s	Capital	Size	Range Area	Inputs Change		Outputs Change	
					Medical Costs	Administrative Costs	Clients	Income/Premiums
1	1.1	-160	1.74×10^{-3}	II	-145	-15	0	382
1	1.1	-141	1.74×10^{-3}	II	-126	-15	0	355
1	0.9	-177	1.42×10^{-3}	IV	-162	-15	0	246
1	0.9	-160	1.42×10^{-3}	IV	-145	-15	0	209

in experiments in subsections 6.3.1 and 6.3.2. DMU F, in its turn, behaves like the larger plans A and C, using all Administrative Costs decreasing available and presenting superefficiency in all instances (Table 6.26). It must be said that all these DMU s are already well efficient, respectively $e'_A = 0.87, e'_C = 0.85, e'_F = 0.67$ which is enough to overcome the normalized frontier ($e' = 1$) when subject to *ModelME*.

Table 6.26: Maximization of Efficiency - $DMU_0 \equiv F$.

csw	K_s	Capital	Size	Range Area	Inputs Change		Outputs Change	
					Medical Costs	Administrative Costs	Clients	Income/Premiums
1	1.1	-76	4.09×10^{-4}	II	-62	-14	0	168
1	1.1	-65	4.09×10^{-4}	II	-51	-14	0	146
1	0.9	-85	3.35×10^{-4}	IV	-71	-14	0	109
1	0.9	-76	3.35×10^{-4}	IV	-62	-14	0	93

When $DMU_0 \equiv C1$, on the other hand, mostly because it is csw -efficient, *ModelME* is lead to similar circumstances of other models, which is, to use also other output variable Clients in order to get the best efficiency results, that is, increasing the ratio of Income/Premiums to Clients. As in other DMU 's, all instances present superefficiency (Table 6.27).

Table 6.27: Maximization of Efficiency- $DMU_0 \equiv C1$.

csw	K_s	Capital	Size	Range Area	Inputs Change		Outputs Change	
					Medical Costs	Administrative Costs	Clients	Income/Premiums
1	1.1	-7.5	7.30×10^{-6}	II	-5.5	-2	-1.35	28
1	1.1	-6.4	7.30×10^{-6}	II	-4.4	-2	-1.25	25
1	0.9	-8.5	6.00×10^{-6}	II	-6.5	-2	-1.89	18
1	0.9	-7.5	6.00×10^{-6}	II	-5.5	-2	-1.8	16

The next experiment presents very singular observations. First, DMU K does not increase its relative size without worsening csw -efficiency. It is needed to say that this is the smallest DMU that, in reality, must faces the biggest challenges to survive. In this case, *ModelME* says that, for DMU K size increase with capital (budget) saving, it is required to invest in operational, sales and marketing structures, which again, is a good guidance for an executive decision. This action can even support Medical Costs decreasing. In the two last instances the efficiency grows, with superefficiency in the third one (Table 6.28).

DMU N presents the worst csw -efficiency and, because of this, it is the only DMU that does not get the superefficiency. Besides, for the reason it is a small health plan, it

Table 6.28: Maximization of Efficiency- $DMU_0 \equiv K$.

csw	K_s	Capital	Size	Range Area	Inputs Change		Outputs Change	
					Medical Costs	Administrative Costs	Clients	Income/Premiums
0.53	1.1	-2	7.00×10^{-7}	I	-3.6	1.6	0	0
0.53	1.1	-1.6	7.00×10^{-7}	I	-3	1.4	0	0
1	0.9	-2.4	7.00×10^{-7}	IV	-1.4	-1.4	0	3
0.99	0.9	-2	7.00×10^{-7}	IV	-1	-1	0	2.5

presents a anomaly already said, in which Administrative Costs overcome Medical Costs. In last instance *ModelME* says that in order to decrease in both relative size and efficiency, *DMU N* must operate with higher level of Medical Costs, saving Administrative Costs (Table 6.29).

Table 6.29: Maximization of Efficiency - $DMU_0 \equiv N$.

csw	K_s	Capital	Size	Range Area	Inputs Change		Outputs Change	
					Medical Costs	Administrative Costs	Clients	Income/Premiums
0.33	1.1	-1.2	3.10×10^{-6}	II	-0.5	-0.7	-0.76	2.5
0.3	1.1	0.26	3.10×10^{-6}	II	0.7	-0.4	-0.5	1.6
0.25	0.9	-1.2	2.60×10^{-6}	III	-1.2	0	0	-1
0.29	0.9	0.26	2.60×10^{-6}	IV	2.3	-2	0	0

At the end, one can say that, in most experiments, single DMU management of capitals models work properly, with strong consistence among all parameters and results, i.e. capital (budget), relative size and *csw*-efficiency. Maximization of efficiency, in turn, shows higher results for efficiency, which was already explained in this subsection. The outcome analysis also leads to reasonable managerial decisions for health plans sector, considering always the amount of input variables (Medical and Administrative Costs), their weights in inferred production function and *DMU's* relative sizes.

6.3.4 *DMUs* Subset - Minimization of Total Capital

The following experiments involve the effort to apply managerial decisions regarding whereas they must be performed not only by one but by a subset of *DMUs*. In this sense, there are two models: first, aiming to minimize *DMUs* subset budget to meet a general *DMU's* efficiency target, computed through the Euclidean distance from a *csw*-efficiencies vector to a ℓ_2 -efficiencies one; second, trying to minimize this Euclidean distance subject to a given budget. Both models admit two different approaches, either considering all *DMUs* (*Altruistic*) in efficiency issue or just those ones from a chosen *DMU's* subset (*Selfish*). In all experiments all *DMUs* are not supposed to be smaller ($t_d \geq 1$).

In this subsection the minimization of capital is experimented in two opposite situations, with worsening of efficiency and, after, with improving. For each one, both

approaches are tested.

Thus, the altruistic minimization of capital with efficiency worsening of 10% (Table 6.30) implies in change of *DMUs* A and C, the largest health plans. *ModelMTC* suggests a total budget increase of 90.59 monetary units, distributed on 1.37 to A and 89.22 to C, which does not promote so much difference to the former but strong transformation to the latter, both in relative size (increasing) and efficiency (worsening). This model points out that *DMUs* must concentrate on let Medical Costs to a little growth in order to meet the general efficiency demand. It must be a possible situation in a scenario in which physicians representatives are claiming to wage rising (what happens every year).

Table 6.30: Minimization of Total Capital - Altruistic ($t_d = 1.1, t_j = 1, c = 90.59$)

<i>DMU</i>	c_j	Size	csw	Range Area	Inputs Change		Outputs Change	
					Medical Cost	Administrative Costs	Clients	Income/Premiums
A	1.37	1	0.87	-	2	-0.6	0	0
C	89.22	1.83×10^{-3}	0.73	I	89	0	0	0
F	0	3.72×10^{-4}	0.67	-	0	0	0	0
C1	0	6.70×10^{-6}	1	-	0	0	0	0
K	0	8.00×10^{-7}	0.58	-	0	0	0	0
N	0	2.90×10^{-6}	0.26	-	0	0	0	0

The same scenario in a selfish approach leads to a budget saving, since *ModelMTC* returns only 0.63 as total capital. In a comparison with previous approach one can note that *DMU* A remains affected but *DMU* K, the smallest health plan, assumes a new role, with changes in relative size and efficiency (Table 6.31).

Table 6.31: Minimization of Total Capital - Selfish ($t_d = 1.1, t_j = 1, c = 0.63$)

<i>DMU</i>	c_j	Size	csw	Range Area	Inputs Change		Outputs Change	
					Medical Cost	Administrative Costs	Clients	Income/Premiums
A	-1.37	1	0.87	-	-2	0.6	0	0
C	-0.05	1.58×10^{-3}	0.85	-	-0.07	0.02	0	0
F	-0.02	3.72×10^{-4}	0.67	-	-0.03	0.01	0	0
C1	-0.002	6.70×10^{-6}	1	-	-0.03	0.001	0	0
K	2	9.00×10^{-7}	0.51	I	2	0	0	0
N	0	2.90×10^{-6}	0.26	-	0	0	0	0

The next two experiments deal with efficiency improving of 10% and both behaves similarly, with movements just in *DMU* N, with the worst efficiency. In first, Table 6.32, *DMU* N needs to save 1.31 of its operational budget to reach the general improvement efficiency goal, focusing on reduce Administrative Costs and increase Income/Premiums. Interesting that, when it is about efficiency improvement *ModelMTC* prefers to use those more weighed variables.

Regarding to selfish approach, only 0.63 of capital saving is required to achieve the efficiency goal (Table 6.33).

Table 6.32: Minimization of Total Capital - Altruistic ($t_d = 0.9, t_j = 1, c = -1.31$)

<i>DMU</i>	c_j	Size	csw	Range Area	Inputs Change		Outputs Change	
					Medical Cost	Administrative Costs	Clients	Income/Premiums
A	0	1	0.87	-	0	0	0	0
C	0	1.58×10^{-3}	0.85	-	0	0	0	0
F	0	3.72×10^{-4}	0.67	-	0	0	0	0
C1	0	6.70×10^{-6}	1	-	0	0	0	0
K	0	8.00×10^{-7}	0.58	-	0	0	0	0
N	-1.31	2.90×10^{-6}	0.32	II	0	-1.31	0	1.44

Table 6.33: Minimization of Total Capital - Selfish ($t_d = 0.9, t_j = 1, c = -0.63$)

<i>DMU</i>	c_j	Size	csw	Range Area	Inputs Change		Outputs Change	
					Medical Cost	Administrative Costs	Clients	Income/Premiums
A	0	1	0.87	-	0	0	0	0
C	0	1.58×10^{-3}	0.85	-	0	0	0	0
F	0	3.72×10^{-4}	0.67	-	0	0	0	0
C1	0	6.70×10^{-6}	1	-	0	0	0	0
K	0	8.00×10^{-7}	0.58	-	0	0	0	0
N	-0.63	2.90×10^{-6}	0.29	II	0	-0.63	0	0.65

6.3.5 *DMUs* Subset - Maximization of Global Efficiency

These experiments deal firstly, with availability of capital and after, its reduction. Thus, considering a positive capital of 100 monetary units, both approaches altruistic (Table 6.34) and selfish (Table 6.35) perform exactly in the same way, showing that there are no effects in focusing just a subset of *DMUs*.

Table 6.34: Maximization of Efficiency - Altruistic ($t_s = 1, t_j = 1, \bar{c} = 100$)

<i>DMU</i>	c_j	Size	csw	Range Area	Inputs Change		Outputs Change	
					Medical Cost	Administrative Costs	Clients	Income/Premiums
A	-182	1	0.55	I	-335	153	0	1011
C	25	1.95×10^{-3}	0.62	I	25	0	0	157
F	47	6.47×10^{-4}	0.57	I	47	0	0	154
C1	0	6.70×10^{-6}	0.59	I	0	0	0	0
K	0	8.00×10^{-7}	0.35	I	0	0	0	0
N	210	5.86×10^{-4}	1	II	210	0	0	493

All *DMUs* get worse about *csw*-efficiencies and present a reduction in their relative sizes, except *DMU* N, that performs oppositely, becoming *csw*-efficient ($e' = 1$) and displaying an expressive growth in its relative size, seeming that *ModelMGE* puts all the efforts to make the worst *DMU* better.

In situations of reduction of capital (-100), both altruistic and selfish approaches perform equally. Once again keeping all efforts in *DMU* N (Tables 6.36 and 6.37).

6.3.6 *DMUs* Subset - Superefficiency

While subsections 6.3.4 and 6.3.5 cannot access superefficiency, the present handle this situation. For short, only Minimization of Capital model is used and the scenarios of

Table 6.35: Maximization of Efficiency - Selfish ($t_s = 1, t_j = 1, \bar{c} = 100$)

DMU	c_j	Size	csw	Range Area	Inputs Change		Outputs Change	
					Medical Cost	Administrative Costs	Clients	Income/Premiums
A	-182	1	0.55	I	-335	153	0	1011
C	25	1.95×10^{-3}	0.62	I	25	0	0	157
F	47	6.47×10^{-4}	0.57	I	47	0	0	154
C1	0	6.70×10^{-6}	0.59	I	0	0	0	0
K	0	8.00×10^{-7}	0.35	I	0	0	0	0
N	210	5.86×10^{-4}	1	II	210	0	0	493

Table 6.36: Maximization of Efficiency - Altruistic ($t_s = 1, t_j = 1, \bar{c} = -100$)

DMU	c_j	Size	csw	Range Area	Inputs Change		Outputs Change	
					Medical Cost	Administrative Costs	Clients	Income/Premiums
A	-182	1	0.74	I	-335	153	0	1011
C	14	1.81×10^{-3}	0.8	I	14	0	0	117
F	25	5.17×10^{-4}	0.69	II	25	0	0	95
C1	0	6.70×10^{-6}	0.8	I	0	0	0	0
K	-0.37	8.00×10^{-7}	0.48	I	-0.37	0	0	0
N	43	5.05×10^{-5}	1	II	43	0	0	115

Tables 6.30 and 6.31 are reproduced, now letting the $DMUs$ of a chosen subset to reach their superefficiencies.

As consequence, efficient $DMUs$ A and C1 present superefficiency, with scores, respectively, of 1.03 and 1.13. In these cases, Algorithm 2 (Section 5.3) is required.

The altruistic approach does not determine any changes for $DMUs$ C1 and N and, as expected, indicates a total budget saving of -1.32 (Table 6.38), even with a worsening in general efficiency score. This shows a different result from *ModelMTC* in which an additional capital expenditure is demanded (Table 6.38).

When the selfish approach is used, the capital saving is more expressive (-1140), highly supported by DMU A. Many different movements can be observed. While DMU A goes towards area II, with strong increase in its efficiency (keeping its *maximum reference size*), other $DMUs$ present an important decrease in their efficiency (displacements to area I), except DMU C1 (csw -efficient) which keeps its status.

Table 6.37: Maximization of Efficiency - Selfish ($t_s = 1, t_j = 1, \bar{c} = -100$)

DMU	c_j	Size	csw	Range Area	Inputs Change		Outputs Change	
					Medical Cost	Administrative Costs	Clients	Income/Premiums
A	-182	1	0.74	I	-335	153	0	1011
C	14	1.81×10^{-3}	0.8	I	14	0	0	117
F	25	5.17×10^{-4}	0.69	II	25	0	0	95
C1	0	6.70×10^{-6}	0.8	I	0	0	0	0
K	-0.37	8.00×10^{-7}	0.48	I	-0.37	0	0	0
N	43	5.05×10^{-5}	1	II	43	0	0	115

Table 6.38: Superefficiency/*ModelSE* - Altruistic ($t_d = 1.1, t_j = 1, c = -1.32$)

<i>DMU</i>	c_j	Size	csw	Range Area	Inputs Change		Outputs Change	
					Medical Cost	Administrative Costs	Clients	Income/Premiums
A	-1.37	1	0.83	I	-2	0.6	0	0
C	-0.05	1.58×10^{-3}	0.81	I	-0.07	0.02	0	0
F	-0.02	3.72×10^{-4}	0.64	I	-0.03	0.01	0	0
C1	0.49	6.70×10^{-6}	1	-	0	0.49	0	0
K	-0.37	8.00×10^{-7}	0.6	II	0	-0.37	0	0
N	0	2.90×10^{-6}	0.25	-	0	0	0	0

Table 6.39: Superefficiency/*ModelSE* - Selfish ($t_d = 1.1, t_j = 1, c = -1140$)

<i>DMU</i>	c_j	Size	csw	Range Area	Inputs Change		Outputs Change	
					Medical Cost	Administrative Costs	Clients	Income/Premiums
A	-1101	1	0.91	II	-1100	-1	6149	-1
C	-30	1.58×10^{-3}	0.84	I	-30	0	0	32
F	-14	3.72×10^{-4}	0.65	I	-14	0	0	12
C1	-0.63	6.70×10^{-6}	1	IV	0	-0.63	0	2.5
K	-0.59	8.00×10^{-7}	0.56	I	-0.59	0	0	0.43
N	6	2.90×10^{-6}	0.18	I	6	0	0	1.6

6.4 Comments

Finally, in general, management of capital models work properly in this application, guiding managerial strategies that are lined with expected decisions in Brazilian health plans. For subset of *DMUs*, *ModelMGE* does not show differences between *Altruistic* and *Selfish* approaches, which needs to be investigated further. There is not any situation that a result invalidates another one and, even considering the meaningfulness of the *DMU's* sample which presents extreme situations on observations, one can conclude that such models could be admitted to enhance the management tools that support decision making process.

Chapter 7

Conclusions and Future Directions

In this work we considered efficiency scores with a common set of weights (*csw-efficiencies*) and introduced the concepts of *relative area*, *relative size* and *range area*. The preliminary idea was to define a comparative measure of size based on virtual inputs and outputs. In addition, we evaluated efficiency and size of *DMUs* on a number of problems regarding management of capital. Essentially, the analysis focused on different combinations of parameters denoting total budget, efficiency scores (with common set of weights) and relative size. In all cases, a manager could investigate alternatives through different optimization models and select the best strategy to be adopted.

The concept of relative area relates virtual inputs with costs and virtual outputs and was used in the definition of relative size. In this way, larger areas are associated with larger relative sizes and vice versa. Despite of good computational results obtained for many different solvers, the definition of relative size introduces a bi-linear function which can produce non-convex models. In order to avoid that, an interesting possibility of research is to deal with other definitions of relative size also based on virtual inputs and outputs. For instance, we can consider $\bar{d}_j = ((\bar{a}_j)^2 + (\bar{b}_j)^2)^{1/2}$ for $j = 1, \dots, n$ (relative distance) and then the normalized distance $\tilde{s}_j = \bar{d}_j / \bar{d}$ with $\bar{d} = \max\{\bar{d}_j, \forall j = 1, \dots, n\}$ (relative size). Note that \bar{d}_j express the Euclidean distance between coordinates $(0, 0)$ and (\bar{a}_j, \bar{b}_j) . The main drawback of this definition is that the virtual outputs are always less or equal than their corresponding virtual inputs (*i.e.*, $\bar{b}_j \leq \bar{a}_j$). As a consequence, *DMUs* with very large outputs and small inputs can correspond to virtual outputs limited above by small virtual inputs. In this case, the coordinates (\bar{a}_j, \bar{b}_j) would be very close to the origin $(0, 0)$ and we would have an unwanted small relative size measure. In order to overcome this hurdle, a possible alternative is to consider the *ModelQP* (3.5)-(3.7) and deal with *DEA* super-efficiencies instead of *DEA* efficiencies computed for all n *DMUs*

in the set. After that, we use Theorem 1 to move all *DMUs* inside the range area and reevaluate their relative distances (or relative sizes).

In chapters 4 and 5, we considered several questions regarding management of capital and many possibilities were explored. There were proposed *two* main approaches. In the former, goals were settled to be pursued by a single *DMU*, whereas in the latter, the goals were set out collectively for a subset Ψ of *DMUs*. It is important to remark that, even for $|\Psi| = 1$, both approaches are slightly different. In the first (chapter 4), we defined goals in the constraints allowing coordinates out of range area, while in the second (Chapter 5), all displacements are inside the range area and goals were stated by a reference vector of *DEA* efficiencies. In a second moment, in chapter 5, if additional targets are established for all *DMUs* of Ψ , then a new reference vector of *DEA* super-efficiencies can be considered to allow movements out of the range area. We also considered two possible situations, namely, *i*) the *altruistic approach* in which all *DMUs* of Ψ help the entire set of *DMUs*; and *ii*) the *selfish approach* in which *DMUs* of Ψ cooperate to help themselves.

Note in chapter 5 that two main optimization problems were considered, namely, *Minimization of the Total Capital* and *Maximization of Global efficiencies*. Thus, an interesting question to be addressed in the future is to consider the maximization of relative size of *one* or more *DMUs* using a target vector of relative sizes (possibly, a *DEA* vector of relative sizes). In this case, in addition to the proposed models for management of capital, a new optimization problem could be defined which objective should be maximize the relative size over Ψ , and it is also subjected to fixed targets of capital and efficiency scores.

A usual problem present in *DEA* is to establish a fair comparison among *DMUs* with very different attributes. For instance, how to compare efficient *DMUs* with small relative size before less efficient *DMUs* with bigger relative sizes? Thus, a possible future work is deal with clustering techniques (see for instance [33],[17],[15]) to better classify the *DMUs* attributes relating budget, efficiencies and relative sizes. As an additional topic of research, it could be worthwhile idea to determine a performance measure combining both efficiency and relative size. For instance, it could be assumed a *performance measure* p_j of a *DMU_j* by setting $p_j = c'e_j + c''s_j$ with $c' + c'' = 1$ and $c', c'' > 0$ denoting, respectively, the importance of efficiency and relative size in the performance.

The results obtained in a Brazilian health plans application are consistent with possible managerial decisions, having the models decided about strategies which would be considered in an actual scenario. Since all inputs present the same costs (A_{ij}), *ModelMC*

(Minimization of Capital for a single *DMU*) consistently searches for reducing Medical Costs in all instances, which would be a reasonable decision, since this is the largest costs center of a health plan and may provide bigger savings. In a very small and inefficient health plans, additionally, this model suggests that some savings must be applied either to Administrative Costs, which would be suitable since this input variable is more relevant in such scenario. *ModelMS* (Maximization of Size for a single *DMU*) is absolutely correspondent to *ModelMC*, so that the size parameter of the former is the outcome of the latter, and that is a result that shows a strong relation between them, which is expected considering the consistency. *ModelME* (Maximization of Efficiency for a single *DMU*), on the other hand, presents a diverse behavior, supporting efficiency increase too regarding the goals in Administrative Costs inputs (saving) and Income/Premiums outputs (increasing). Indeed, these are the two main areas to be addressed in a health plan when an efficiency issue comes up and produces the most intense effects. Once again, the models respond accordingly with a market typical decision. A special fact that differs, in all these cases, is that the models are able to point out an objective goal.

The choice of altruistic or selfish approaches does not interfere in worsening the overall efficiency with *Model MTC*. As expected, the final cost demanded for new overall efficiency score is larger in the former than in the latter, dealing, respectively, with Medical Costs (larger amounts, lower weight) and Administrative Costs (smaller amounts, greater weight). On the other hand, when overall efficiency must be improved, in both altruistic or selfish approach, the health plan with lowest efficient is picked and its Administrative Costs inputs is reduced, which is a decision expected to be made. When dealing with maximization of global efficiency (*ModelMGE*), regardless the approach, the same results are found and based on a general Income/Premiums increase, that is one of the first attempt in health plans market. Finally, in a superefficiency situation applied to *ModelMTC*, (*ModelSE₁*) responds with smaller amounts of capital.

These results encourage in the sense that presented concepts and management of capital models may be easily used through *DEACM*, an accessible tool that is available to small and medium organizations, in which are demanded only the knowledge of their own business. Finally, the present work carries the purpose to experiment new possibilities and it is also interested in supporting the search for new applications for *DEA*.

References

- [1] ALMEIDA, M. R. *A eficiência dos investimentos do Programa de Inovação Tecnológica em Pequena Empresa (PIPE): uma integração da análise envoltória de dados e índice Malmquist*. 2010. PhD thesis, Tese (Doutorado em Engenharia de Produção). Escola de Engenharia de São Carlos. Universidade de São Paulo, 2010.
- [2] ANDERSEN, P.; PETERSEN, N. C. A procedure for ranking efficient units in data envelopment analysis. *Management Science* 39 (10) (1993), 1261–1264.
- [3] ANS. Atlas econômico-financeiro da saúde suplementar 2014, Oct. 2016.
- [4] BANKER, R. D.; CHARNES, A.; COOPER, W. W. Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management science* 30, 9 (1984), 1078–1092.
- [5] BROCKETT, P. L.; CHANG, R. E.; ROUSSEAU, J. J.; SEMPLÉ, J. H.; YANG, C. A COMPARISON OF HMO EFFICIENCIES FUNCTION OF PROVIDER AUTONOMY. *The Journal of Risk and Insurance* 71, 1 (2004), 1–19.
- [6] BRYCE, C. L.; ENGBERG, J. B.; WHOLEY, D. R. Methods Comparing the Agreement Among Alternative Models in Evaluating HMO Efficiency. *Health Services Research* (1994), 509–528.
- [7] CARRINGTON, R.; COELLI, T.; PRASADA RAO, D. Australian private health insurance productivity growth: Is there scope to limit premium increases?*. *Economic Record* 87, 276 (2011), 125–139.
- [8] CHARNES, A.; COOPER, W.; RHODES, E. Measuring the efficiency of decision making units. *European Journal of Operational Research* 2, 6 (Nov. 1978), 429–444.
- [9] CHARNES, A.; OF TEXAS AT AUSTIN. CENTER FOR CYBERNETIC STUDIES, U. *Foundations of Data Envelopment Analysis for Pareto-Koopmans Efficient Empirical Production Functions*. Research report (University of Texas at Austin. Center for Cybernetic Studies). Center for Cybernetic Studies, University of Texas at Austin, 1985.
- [10] COOPER, W. W.; SEIFORD, L. M.; TONE, K. *Introduction to data envelopment analysis and its uses: with DEA-solver software and references*. Springer Science & Business Media, 2006.
- [11] COOPER, W. W.; SEIFORD, L. M.; TONE, K. *Data envelopment analysis: a comprehensive text with models, applications, references and DEA-solver software*. Springer Science & Business Media, 2007.

- [12] DOYLE, J. Multiattribute choice for the lazy decision maker: let the alternatives decide. *Organizational Behav. Hum. Decision Process* 62 (1995), 87–100.
- [13] DYSON, R. G.; ALLEN, R.; CAMANHO, A. S.; PODINOVSKI, V. V.; SARRICO, C. S.; SHALE, E. A. Pitfalls and protocols in dea. *European Journal of operational research* 132, 2 (2001), 245–259.
- [14] FERNANDES, E.; PIRES, H. M.; IGNACIO, A. A. V.; SAMPAIO, L. M. D. An analysis of the supplementary health sector in Brazil. *Health policy (Amsterdam, Netherlands)* 81, 2-3 (May 2007), 242–57.
- [15] FLOKOU, A.; KONTODIMOPOULOS, N.; NIAKAS, D. Employing post-dea cross-evaluation and cluster analysis in a sample of greek nhs hospitals. *Journal of Medical Systems* 35, 5 (2011), 1001–1014.
- [16] FRIEDMAN, L.; SINUANY-STERN, Z. Scaling units via the canonical correlation analysis in the dea context. *European Journal of Operational Research* 100 (1997), 629–637.
- [17] GHOLAM R. AMIN, ALI EMROUZNEJAD, S. R. Some clarifications on the dea clustering approach. *European Journal of Operational Research* 215 (2011), 498–501.
- [18] GOLANY, B.; ROLL, Y. An application procedure for DEA. *Omega* 17, 3 (Jan. 1989), 237–250.
- [19] GSTACH, D. Another approach to data envelopment analysis in noisy environments: Dea+. *Journal of Productivity Analysis* 9, 2, 161–176.
- [20] HSU, F.-M.; HSUEH, C.-C. Measuring relative efficiency of government-sponsored r&d projects: A three-stage approach. *Evaluation and program planning* 32, 2 (2009), 178–186.
- [21] JENKINS, L.; ANDERSON, M. A multivariate statistical approach to reducing the number of variables in data envelopment analysis. *European Journal of Operational Research* 147, 1 (2003), 51–61.
- [22] KAO, C. Efficiency decomposition in network data envelopment analysis: A relational model. *European Journal of Operational Research* 192, 3 (2009), 949–962.
- [23] KAO, C.; HUNG, H.-T. Data envelopment analysis with common weights: the compromise solution approach. *Journal of Operational Research Society* 56 (2005), 1196–1203.
- [24] KEMPF, K. G.; ERHUN, F.; HERTZLER, E. F.; ROSENBERG, T. R.; PENG, C. Optimizing capital investment decisions at intel corporation. *Interfaces* 43, 1 (2013), 62–78.
- [25] KONTODIMOPOULOS, N.; NIAKAS, D. Efficiency measurement of hemodialysis units in Greece with data envelopment analysis. *Health Policy* 71, 2 (2005), 195–204.

- [26] LI, S.; JAHANSHAHLOO, G. R.; KHODABAKHSHI, M. A super-efficiency model for ranking efficient units in data envelopment analysis. *Applied Mathematics and Computation* 184 (2007), 638–648.
- [27] LINNA, M. Measuring hospital cost efficiency with panel data models. *Health economics* 7, 5 (1998), 415–427.
- [28] LIU, J. S.; LU, L. Y.; LU, W.-M.; LIN, B. J. A survey of DEA applications. *Omega* 41, 5 (Oct. 2013), 893–902.
- [29] MEHRABIAN, S.; ALIREZAEI, A.; JAHANSHAHLOO, G. A complete efficiency ranking of decision making units in dea. *Computational Optimization and Applications* 14 (1999), 261–266.
- [30] MILLIKEN, O.; DEVLIN, R. A.; BARHAM, V.; HOGG, W.; DAHROUGE, S.; RUSSELL, G. Comparative efficiency assessment of primary care service delivery models using data envelopment analysis. *Canadian Public Policy* 37, 1 (2011), 85–109.
- [31] MOREY, R.; FINE, D.; LOREE, S. Comparing the allocative efficiencies of hospitals. *Omega* 18, 1 (Jan. 1990), 71–83.
- [32] MOURA, A. L. D.; RUBEM, A. P. D. S.; MELLO, J. C. C. B. S., ET AL. Avaliação de empresas aéreas com modelo de análise envoltória de dados clusterizado pela escala de operação. *Journal of Transport Literature* 10, 1 (2016), 20–24.
- [33] NAJADAT, H.; NYGARD, K. E.; SCHESVOLD, D. Clustering-based method for data envelopment analysis. *North Dakota State University, Technical Report* (2005).
- [34] ROLL, Y.; GOLANY, B. Alternate methods of treating factor weights in dea. *Omega* 21 (1993), 99–109.
- [35] ROLLINS, J.; LEE, K.; XU, Y.; OZCAN, Y. A. Longitudinal study of health maintenance organization efficiency. *Health Services Management Research* 14, 4 (2001), 249–262.
- [36] ROSENMAN, R.; SIDDHARTHAN, K.; AHERN, M. Output efficiency of health maintenance organizations in Florida. *Health economics* 6, 3 (1997), 295–302.
- [37] SALINAS-JIMÉNEZ, J.; SMITH, P. Data envelopment analysis applied to quality in primary health care. *Annals of Operations Research* 67, 1 (1996), 141–161.
- [38] SAMPAIO, L. M. D.; DE, M. J. F. An evaluation of the managed care program as applied to the brazilian supplementary health service. In *Operational Research for Health Policy: Making Better Decisions: Proceedings of the 31st Annual Conference of the European Working Group on Operational Research Applied to Health Services* (2007), Lang, Peter, AG, Internationaler Verlag Der Wissenschaften, p. 191.
- [39] SIDDHARTHAN, K.; AHERN, M.; ROSENMAN, R. Data envelopment analysis to determine efficiencies of health maintenance organizations. *Health care management science* 3, 1 (Jan. 2000), 23–9.
- [40] TONE, K. A slacks-based measure of super-efficiency in data envelopment analysis. *European Journal of Operational Research* (1) 143 (2002), 32–41.

- [41] WAGNER, J. M.; SHIMSHAK, D. G. Stepwise selection of variables in data envelopment analysis: Procedures and managerial perspectives. *European journal of operational research* 180, 1 (2007), 57–67.
- [42] WANG, E. C.; HUANG, W. Relative efficiency of r&d activities: A cross-country study accounting for environmental factors in the dea approach. *Research Policy* 36, 2 (2007), 260–273.
- [43] WORLD HEALTH ORGANIZATION. World Health Statistics 2015. Tech. rep., Geneva - Switzerland, 2015.
- [44] YANG, Z. A two-stage DEA model to evaluate the overall performance of Canadian life and health insurance companies. *Mathematical and Computer Modelling* 43, 7 (2006), 910 – 919.
- [45] YU, P. A class of solutions for group decision problems. *Management Science* 19 (1973), 936–946.

APPENDIX A - *DEACM* USER'S MANUAL

A.1 GENERAL INFORMATION

A.1.1 System Overview

The *DEACM* stands for Data Envelopment Analysis for Capital Management and is a web application that is intended to help solve problems related to Capital Management. The application is based on a Doctoral Thesis of the researcher Ney Paranaguá de Carvalho that proposed the concept of relative size and show how to use it to solve problems regarding Capital Management.

The Doctoral thesis proposes a series of mathematical models for optimization problems regarding Capital Management, that are difficult, error prone and time consuming to execute them by hand and make sense of the results, specially with a big amount of data. The *DEACM* application implements these mathematical models and make it easy to input data, execute the proposed models and analyze the results.

The major functions performed by the application are:

- **Input or load data:** this feature enables the user to input data using a web interface and manually type in the matrix of *DMU's* (*Decision Making Units*) with it's associated data: inputs, outputs and costs. It's also possible to import a .csv file with the DMUs data along with the capital management model and it's parameters in case of a bigger data;
- **Execute base models (*DEA* and *KAO*):** this feature frees the user for the need of giving the values of the DEA and KAO's efficiency as input arguments for each DMU. This feature saves time and avoid errors in input data;
- **Execute capital management models:** this feature enables the user to select a capital management model, give the parameters for that specific model and executes it. It automates the execution of the mathematical models.

A.2 Acronyms and Abbreviations

- *DEACM* - name of the application and stands for Data Envelopment Analysis for Capital Management
- *DEA* - stands for Data Envelopment Analysis and it's a mathematical model used to measure the efficiency of *DMU's*
- *KAO* - it's a reference to the model proposed by C. Kao and H.-T. Hung for commons weights based on DEA
- *DMU* - stands for Decision Making Unit and it's the name given for each entity with inputs and outputs in the DEA model

A.3 SYSTEM SUMMARY

The main goal of the *DEACM* application is to help the user to solve problems related to capital management. The capital management here is any economic activity carried out by one or more *DMUs* involving saving or investment of resources. To help solving these problems, the application allows the user to preview the output of controlled modifications on the budget, relative size and efficiency scores.

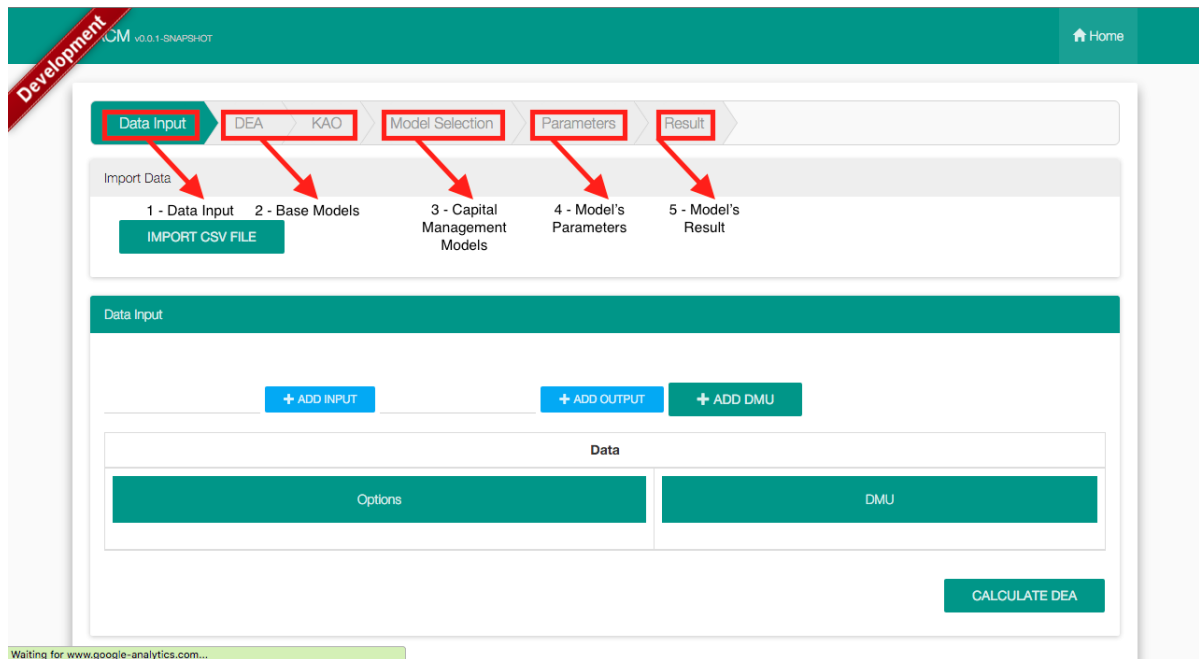
For instance, in the case of a downturn of the economy, some decreasing of the relative size and/or efficiency may be wanted to save capital. In general, motivated by different economic scenarios, a manager could establish a number of objectives and directions for the company and use the application to test and preview different scenarios to achieve those goals based on three parameters relating efficiency, capital and relative size.

A.3.1 Data Flows

The *DEACM* application follows a step-by-step user interaction to perform it's functions, as shown in Figure A.1.

Each step depends of the result of the previous one and it's results are forwarded to the following step. The steps are:

- **Data Input:** this is the first step, where the user manually type in the data or import a *CSV* file;

Figure A.1: *DEACM* Steps

- **Base Models:** it's composed of the two steps, *DEA* and *KAO*, which are the basis for the proposed models for Capital Management. In these step, the application calculates the *DEA* and *KAO* efficiencies for each *DMU*;
- **Model Selection:** in this step, the user selects the Capital Management model to be executed;
- **Parameters:** in this step, the user give the parameters for the chosen Capital Management model;
- **Result:** it's the last step, where the application shows the result of the execution of the selected step with the given parameters.

The details of each step are presented in the following section.

A.4 GETTING STARTED

The *DEACM* application follows a step-by-step style of user interaction where the following step depends on the previous one to execute. The steps, in the corresponding order, are: *Data Input*, *DEA*, *KAO*, *Model Selection*, *Parameters* and *Result*.

Inside each step, it's possible to go back to the previous step by clicking on the step name at the top bar or by clicking at the go back button. The following sections contains

the explanation of each step of the DEACM application and how to use it, along with some special cases.

A.4.1 Data Input

This is the first step of the application where the user enters the data. The data is a matrix where the lines correspond to a single DMU and the columns are inputs, outputs and it's corresponding data. Figure A.2 explain each element, as follows:

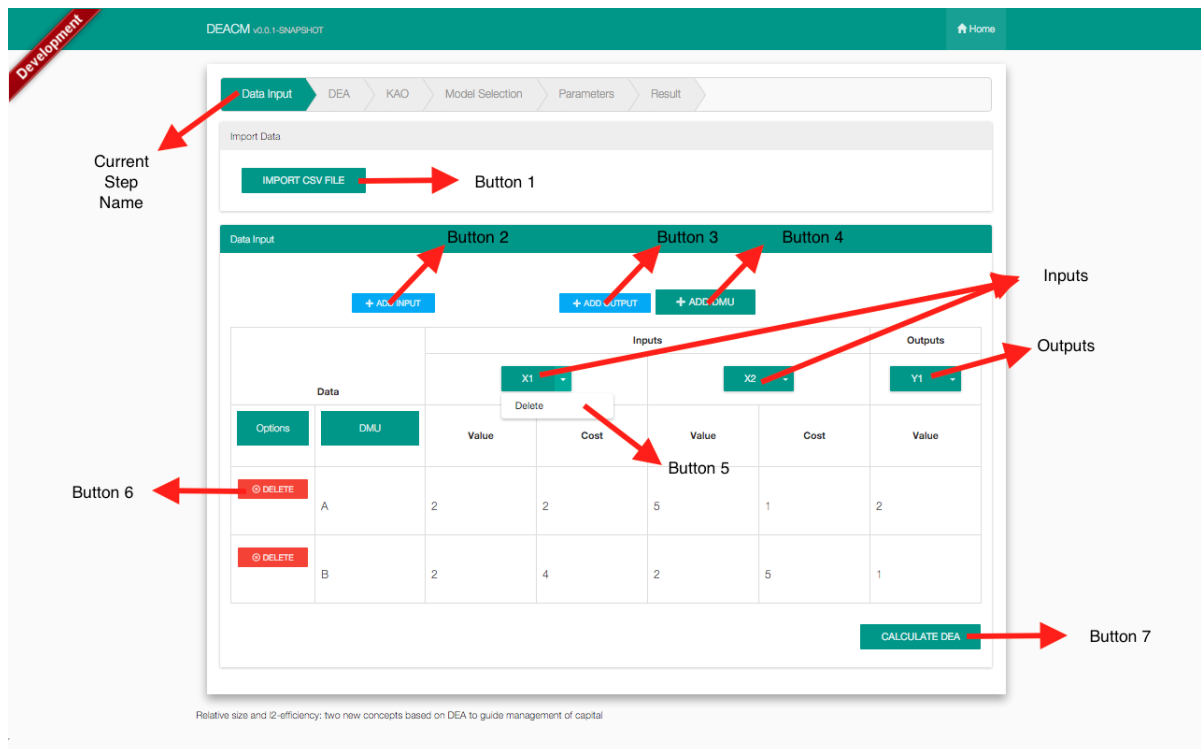


Figure A.2: Data Input

- Button 1: Using this button, the user can import a *CSV* file with the data;
- Button 2: adds a new input to the matrix of data;
- Button 3: adds a new output to the matrix of data;
- Button 4: adds a new *DMU* to data matrix;
- Button 5: deletes the selected input or output;
- Button 6: This button deletes the selected *DMU*;
- Button 7: This button go to the next step which is calculate the *DEA*.

A.4.1.1 Importing data

The application supports the import of a *CSV* file with the input data along with the model to execute and the data associated to the model. To import a *CSV* file, you need to click on the **Button 1 - Import CSV File**, select the *CSV* file and click the **Load** button, as shown in Figure A.3. If the given file is correct, the application load it's contents, automatically executes the *DEA*, *KAO* and Model Selection steps and return to the Parameters step, so the user can change the parameters and execute the model.

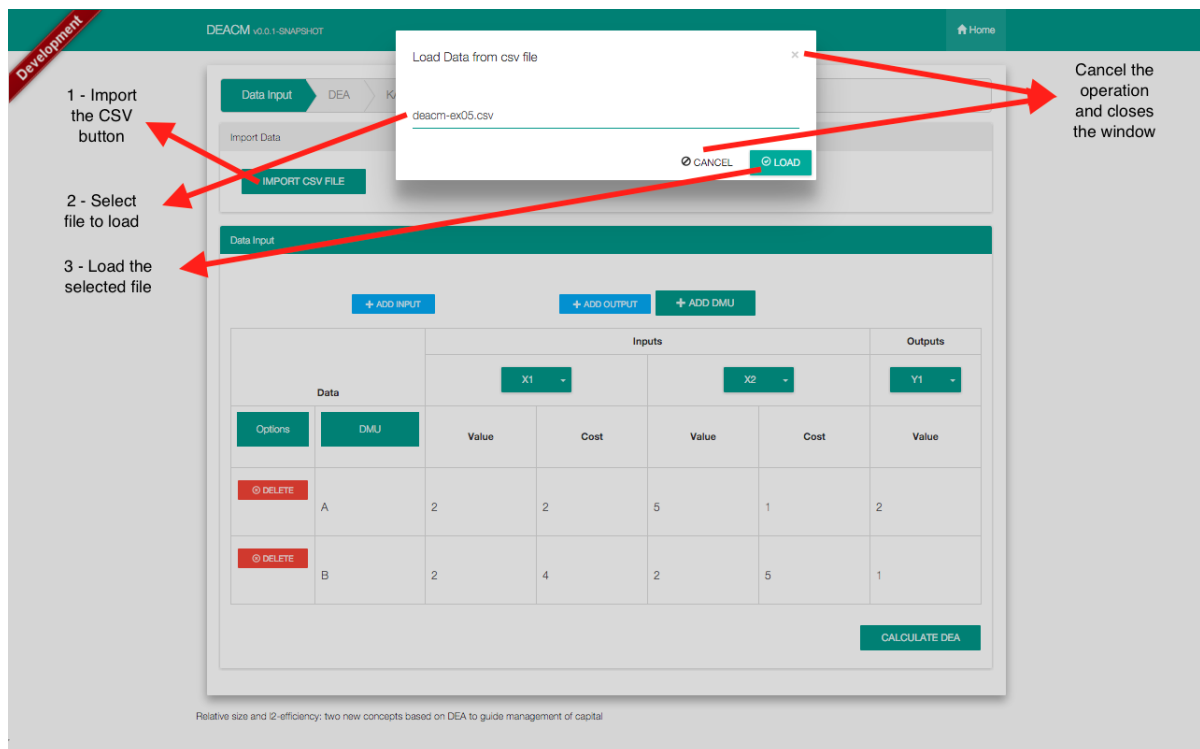


Figure A.3: Data Input

If the file is not in the expected format, the application prompt an error message.

A.4.1.2 Adding an input

To add an input, just type in the name of the input on the input box next to the **Button 2 - Add Input** and click on it. If there is any error, the application will show an error message, otherwise it will add a new column on the table with the given input name.

A.4.1.3 Adding an output

To add an output, just type in the name of the output on the input box next to the **Button 3 - Add Output** and click on it. If there is any error, the application will show

an error message, otherwise it will add a new column on the table with the given output name.

A.4.1.4 Adding an DMU

To add a DMU, just click the **Button 4 - Add DMU**. The application will add a new line on the table with the *DMU* name in blank. Now click on the input box in the new line of the table and type in the *DMU* name.

A.4.1.5 Deleting an input

To delete an input, click on the arrow icon next to the input name in the table and select the option delete. This is shown in the figure as **Button 5**.

A.4.1.6 Deleting an output

To delete an output, click on the arrow icon next to the output name in the table and select the option delete. This is shown in the figure as **Button 5**.

A.4.1.7 Deleting an *DMU*

To delete a *DMU*, click on the **Button 6 - Delete** of the corresponding line of the table data.

A.4.1.8 Go to next step

To go to the next step, just click on the **Button 7 - Calculate DEA**. If there's any error with the given data, then the application will show an error message, otherwise, it will go to the next step **Calculate DEA**.

A.5 *DEA*

This is the second step where the application calculates the first basis model, the *DEA model*. This step do not require any input or action by the user, it just shows the calculated *DEA* efficiency for each *DMU* given in the previous step, so that the user can easily follow the calculations ahead and check the values as shown in Figure A.4.

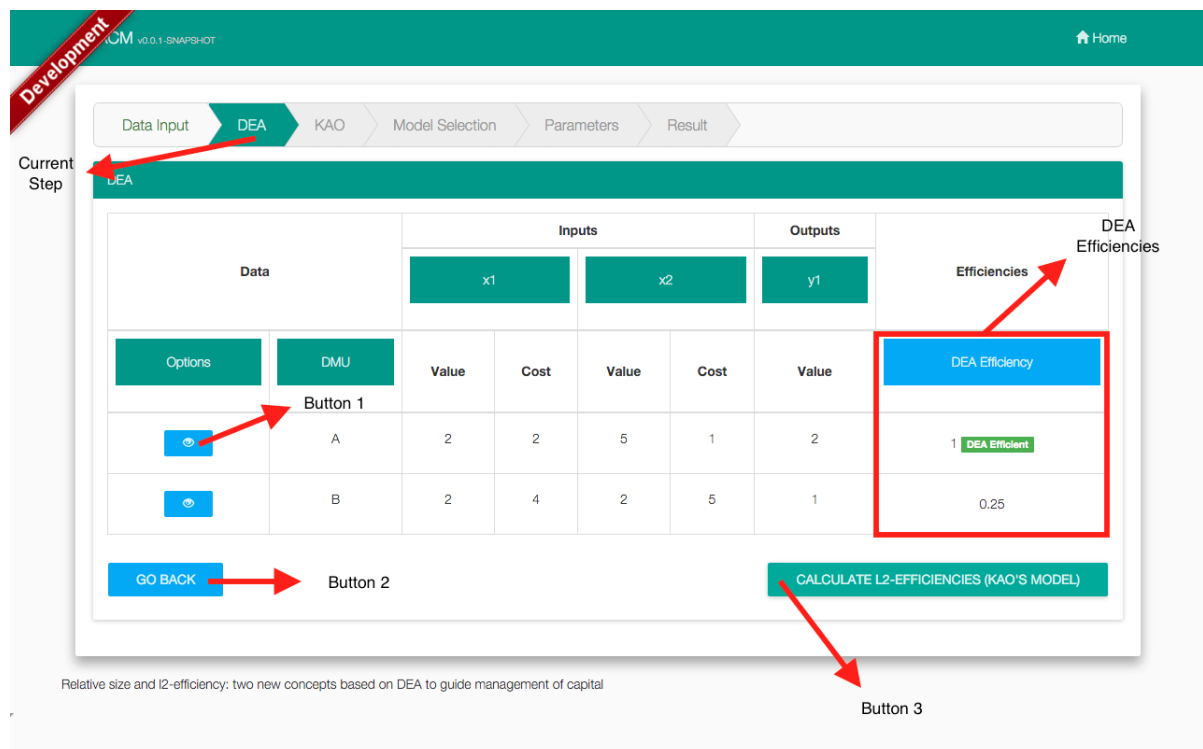


Figure A.4: DEA Computation

- Button 1: Shows the detail of the selected *DMU*
- Button 2: Go back to the previous step
- Button 3: Go to the next step **Calculate ℓ_2 -efficiencies (*KAOs* Model)**

A.5.1 *DMU* Details

To check the *DMU* detail, you just need to click the Button 1 and a modal window will appear with the details of the selected *DMU* as shown in Figure A.5.

A.5.2 Going Back

To go back to the previous step **Data Input**, you just need to click on the **Button 2 - Go Back** or click on the **Data Input** label at the bar with all step names.

A.5.3 Go to next step

To go to the next step *KAO*, you just need to click on the **Button 3 - Calculate ℓ_2 -efficiencies (*KAOs* Model)**.

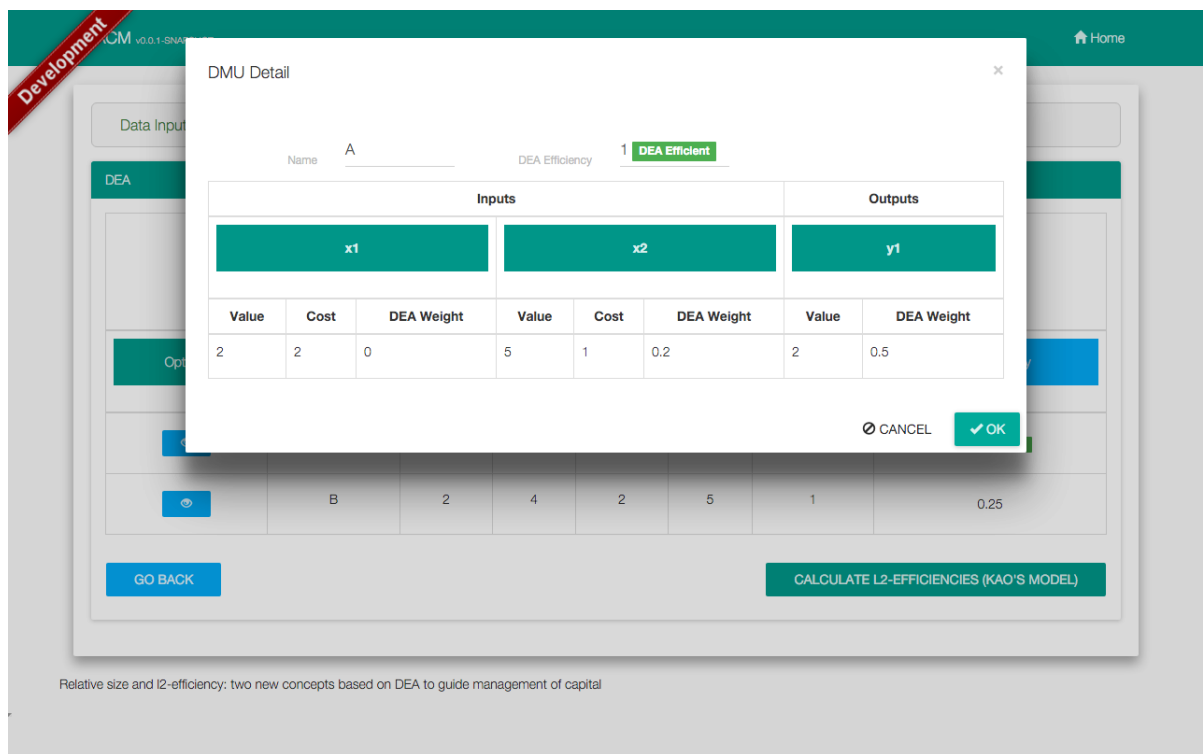


Figure A.5: *DMU* Details

A.6 KAO

This is the third step where the application calculates the second basis model, the KAO model. This step do not require any input or action by the user, it just shows the calculated DEA and KAO efficiencies along with the Relative Size and Relative Area for each DMU given in the previous step, so that the user can easily follow the calculations ahead and check the values as shown in Figure A.6.

A.6.1 *DMU* Details

To check the *DMU* detail, you just need to click the Button 1 and a modal window will appear with the details of the selected *DMU* as shown in Figure A.5.

- Button 1: Shows the detail of the selected *DMU*
- Button 2: Go back to the previous step
- Button 3: Go to the next step **Choose Capital Management Model.**

- for its efficiency. These factors could increase or decrease the *DMUs* size and/or efficiency. The application depends on the user;
- **Maximization of Relative Size:** this model shows how much the relative size of a target *DMU* is likely to grow/shrink according to the available capital and a efficiency factor;
 - **Maximization of Efficiency:** this model shows how much the efficiency of a target *DMU* is likely to grow/shrink according to the available capital and a relative size factor.
- **Subset of *DMUs*:** within this group are those models that are applied to a target subset of *DMUs*. The available models in this group are:
 - **Minimization of Capital:** this model shows the minimum capital that needs to be applied to a target subset of *DMUs* constrained by a factor for the relative size for each *DMU* and a factor for the distance between the *KAO* and *DEA* efficiencies. These factors could increase or decrease the *DMUs* size and/or efficiency. The application depends on the user;
 - **Maximization of Efficiency:** this model shows how much the efficiency of a target subset of *DMUs* is likely to grow/shrink according to the available capital and a relative size factor for each *DMU*.

Figure A.7 explains the screen elements:

A.7.1 Going Back

To go back to the previous step *KAO*, you just need to click on the **Button 1 - Go Back** or click on the ***KAO*** label at the bar with all step names.

A.7.2 Go to next step

To go to the next step, you need to click on the model's name, then the application will take you to the next step according to the model that you've selected.

A.8 Parameters

This is the fifth step where the arguments for the specific model are set. Since the Capital Management models are an optimization problem and fundamentally a mathematical

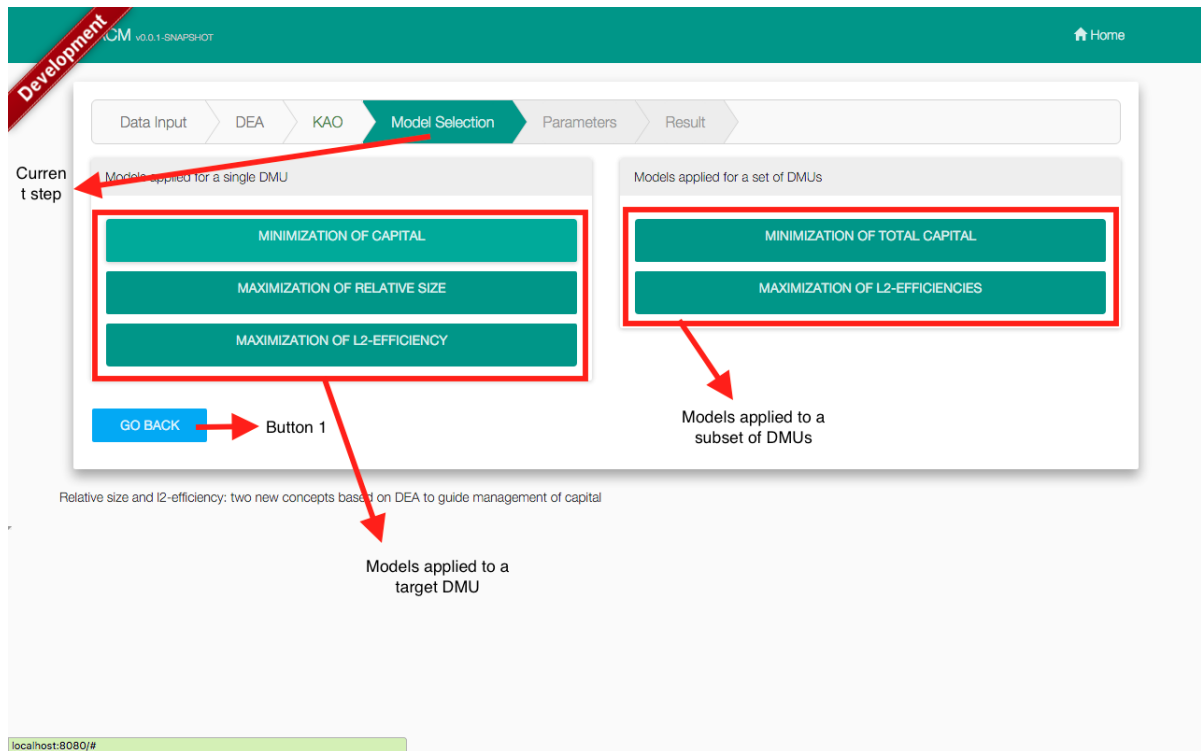


Figure A.7: Model Selection

model, depending on the given arguments, the problem can be infeasible. Another possibility, for all models, is that depending on the arguments, the number of variables and constraints of the model can be too high for the solver give a solution that the problem is considered infeasible. In both cases, the application shows an error message, but the interpretation of the model's argument, it's feasibility and results are up the user.

The following list is a detail of each model, with it's arguments and screen elements explanation.

- **Single *DMU* Models**

- **Minimization of Capital:** this model requires two arguments:

- * **Relative Size Coefficient:** it's a number, representing a factor that will be used to determined the new size of a target *DMU*. For example, the number 0.9 means that the size can be decreased in 10%. The number 1 means the size will be the same and the number 2 means it will be doubled.
 - * **ℓ_2 -efficiency Coefficient:** it's a number, representing a factor that will be used to determined the new *csu*-efficiency of a target *DMU*. For example, the number 0.9 means that the efficiency can be decreased in 10%. The number 1 means the efficiency will be the same and the number 2 means it

will be doubled. This model also requires the user to select a single target *DMU* by clicking on the radio box on the column “target” of the table that contains all the *DMUs*. For the target *DMU*, it’s also required the costs of each input, along with the lower bounds of each input and output. After all parameters are set, you are ready to execute the model. To execute it, just click on the **Button 3 - Execute** on the bottom right of the screen. If there’s any missing information or wrong parameters, the application will show an error message, the same goes if the execution leads to an infeasible problem (Figure A.8).

Develop

Data Input DEA KAO Model Selection **Parameters** Result

Minimize Capital of Single DMU

Relative Size Coefficient 0.9 Relative Size Factor Argument Relative Efficiency Factor Argument L2 Efficiency Coefficient 1.1

Data		Inputs						
		x1			x2			
Options	Target	DMU	Value	Cost	Lower Bound	Value	Cost	Lower Bound
<input checked="" type="radio"/>	<input checked="" type="radio"/>	A	2	2	2	5	1	2
<input type="radio"/>	<input type="radio"/>	B	2			2		

GO BACK EXECUTE

Button 1 Button 2 Button 3

Figure A.8: Single *DMU* - Capital Minimization

- **Button 1:** shows the *DMU* details, just as in the previous steps;
- **Button 2:** goes back to the previous step, just as in the previous steps.
- **Maximization of Relative Size:** this model also requires two arguments:
 - * **Capital:** it’s a number, representing the available capital to invest at the target *DMU*. For example, the number 10, means 10 monetary units available to invest at the target *DMU*;
 - * **l_2 -efficiency Coefficient:** it’s a number, representing a factor that will be used to determined the new *csu*-efficiency of a target *DMU*. For ex-

ample, the number 0.9 means that the efficiency can be decreased in 10%. The number 1 means the efficiency will be the same and the number 2 means it will be doubled.

This model also requires the user to select a single target DMU by clicking on the radio box on the column “target” of the table that contains all the *DMUs*. For the target DMU, it’s also required the costs of each input, along with the lower bounds of each input and output.

After all parameters are set, you are ready to execute the model. To execute it, just click on the **Button 3 - Execute** on the bottom right of the screen. If there’s any missing information or wrong parameters, the application will show an error message, the same goes if the execution leads to an infeasible problem (Figure A.9).

Develop

Data Input DEA KAO Model Selection **Parameters** Result

Maximize Size of Single DMU

Capital 10 Available Capital to invest L2 Efficiency Coefficient 1.1 Relative Efficiency Factor Argument

Data		Inputs						
		x1			x2			
Options	Target	DMU	Value	Cost	Lower Bound	Value	Cost	Lower Bound
<input checked="" type="radio"/>	<input checked="" type="radio"/>	A	2	2	2	5	1	2
<input type="radio"/>	<input type="radio"/>	B	2			2		

GO BACK EXECUTE

Figure A.9: Single *DMU* - Maximization of Relative Size

- **Button 1:** shows the *DMU* details, just as in the previous steps;
- **Button 2:** goes back to the previous step, just as in the previous steps.
- **Maximization of Efficiency:** this model requires two arguments, too:
 - * **Capital:** it’s a number, representing the available capital to invest at the

target *DMU*. For example, the number 10, means 10 monetary units available to invest at the target *DMU*;

- * **Relative Size Coefficient:** it's a number, representing a factor that will be used to determined the new relative size a target *DMU*. For example, the number 0.9 means that the size can be decreased in 10%. The number 1 means the size will be the same and the number 2 means it will be doubled.

This model also requires the user to select a single target *DMU* by clicking on the radio box on the column “target” of the table that contains all the *DMUs*. For the target *DMU*, it's also required the costs of each input, along with the lower bounds of each input and output.

After all parameters are set, you are ready to execute the model. To execute it, just click on the Button 3 - Execute on the bottom right of the screen. If there's any missing information or wrong parameters, the application will show an error message, the same goes if the execution leads to an infeasible problem (Figure A.10).

Development CM v0.0.1-SNAPSHOT Home

Data Input DEA KAO Model Selection **Parameters** Result

Maximize Efficiency of Single DMU

Available Capital to invest: Capital 10

Relative Size Coefficient: 1.2

Relative Size Factor Argument

Data		Inputs						
		x1			x2			
Options	Target	DMU	Value	Cost	Lower Bound	Value	Cost	Lower Bound
Button 1	<input checked="" type="radio"/>	A	2	2	2	5	1	2
<input type="radio"/>	<input type="radio"/>	B	2			2		

GO BACK Button 2 Button 3 EXECUTE

Figure A.10: Single *DMU* - Maximation of Efficiency

- **Button 1:** shows the *DMU* details, just as in the previous steps;
- **Button 2:** goes back to the previous step, just as in the previous steps.

- **Subset of *DMUs* Models:** These models work the same way as the others, but they refer to a target subset of *DMUs*. Both models also have two different ways to execute, which are:

- **DEA Super Efficiency Model:** executing the model in this mode will cause the application to use the *DEA Super Efficiency* for each *DMU* of the target subset. To execute the application with the *DEA Super Efficiency*, just click on the **Execute** button and a pop message will prompt you, as shown in the Figure A.11.

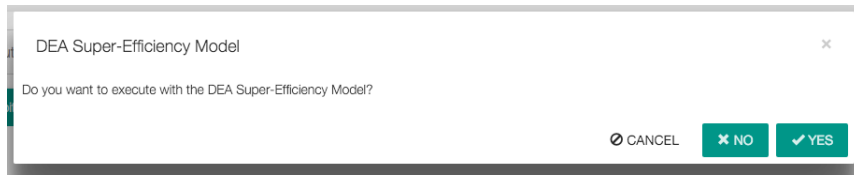


Figure A.11: Subset of *DMUs* - Superefficiency Mode

- **Altruistic or Selfish:** these are the two approaches that the models proposed. In the *altruistic* approach, all *DMUs* of the target subset help the entire set of *DMUs*, while in the *selfish* approach, the *DMUs* of the target subset cooperate to help themselves. To choose between the two approaches, just click on the **Execute** button and a pop message will prompt you, as shown in Figure A.12.

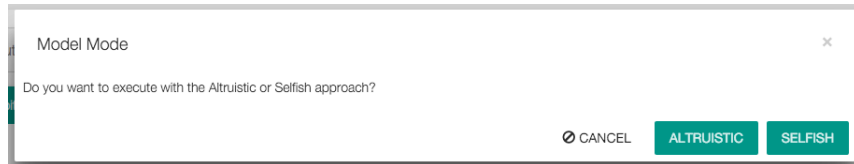


Figure A.12: Subset of *DMUs* - Altruistic or Selfish approaches

- **Minimization of Capital:** this model requires two arguments:
 - * **Update Rating of Distance between ℓ_2 and *DEA* efficiencies:** it's a number, representing a factor that will be used to increase or decrease the distance between the vector of *DEA* and *KAO* efficiencies. For example, the number 0.9 means that distance will be decreased in 10%, or it's expected an overall efficiency improvement. The number 1 means the distance will be kept and the number 2 means it will be doubled;
 - * **Relative Size Coefficient:** it's a number, representing a factor that will be used to determine the smaller relative size admitted to each *DMU* of

the chosen set.

To select a target DMU, just click on the check box on the column *target* of the table that contains all the *DMUs*. For the target *DMUs*, are also required the costs of each input, along with the lower bounds of each input and output.

After all parameters are set, you are ready to execute the model. To execute it, just click on the **Button 3 - Execute** on the bottom right of the screen. If there's any missing information or wrong parameters, the application will show an error message, the same goes if the execution leads to an infeasible problem (Figure A.13).

Develop

Data Input DEA KAO Model Selection **Parameters** Result

Minimize Capital of set of DMUs

Update Rating of Distance between L2 and DEA efficiencies 1.1 → Efficiency Distance Factor

Data				Inputs					
				x1					
Options	Target	DMU	Coefficient of Relative Size	Value	Cost	Lower Bound	Value	Cost	
<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>	A	1	2	2	2	5	1	
<input type="checkbox"/>	<input type="checkbox"/>	B	1	2	1	1	2	1	

GO BACK → Button 2 Button 3 ← EXECUTE

Figure A.13: Subset of *DMUs* - Minimization of Capital

- **Button 1**: shows the *DMU* details, just as in the previous steps;
- **Button 2**: goes back to the previous step, just as in the previous steps.
- **Maximization of Efficiency**: this model requires two arguments:
 - * **Capital**: it's a number, representing the available capital to invest at the target *DMUs*. For example, the number 10, means 10 monetary units available to be invested at the *DMUs* of subset;

*

- * **Relative Size Coefficient:** it's a number, representing a factor that will be used to determine the smaller relative size admitted to each *DMU* of the chosen set.

To select a target *DMU*, just click on the check box on the column *target* of the table that contains all the *DMUs*. For the target *DMUs*, are also required the costs of each input, along with the lower bounds of each input and output.

After all parameters are set, you are ready to execute the model. To execute it, just click on the **Button 3 - Execute** on the bottom right of the screen. If there's any missing information or wrong parameters, the application will show an error message, the same goes if the execution leads to an infeasible problem (Figure A.14).

Develop

Data Input DEA KAO Model Selection **Parameters** Result

Maximize Efficiency of Set of DMUs

Available Capital 10 → Available Capital to invest

Data				Inputs				
				x1				
Options	Target	DMU	Coefficient of Relative Size	Value	Cost	Bound	Value	Cost
<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>	A	1	2	2	2	5	1
<input type="checkbox"/>	<input type="checkbox"/>	B	1	2	1	1	2	1

GO BACK → Button 2 Button 3 ← EXECUTE

Figure A.14: Subset of *DMUs* - Maximization of Efficiency

- **Button 1:** shows the *DMU* details, just as in the previous steps;
- **Button 2:** goes back to the previous step, just as in the previous steps.

A.9 Results

This is the last step where you can see the model's result. Each model gives a different result and the interpretation of the results are up to you.

The result screen shows the model executed, it's arguments, the model's result, each *DMU* details and if two situations occurred during the execution: *relative oversize* and *super-efficiency*. The oversize situation means that, at least, one of the *DMUs* got out of the range area. The same goes to the efficiency.

Figures A.15 to A.19 show the result screen for each model. All the screens share two buttons, the **Go Back** Button and the **Execute Another Model** Button, which takes you to the previous step or execute another model with the same arguments respectively.

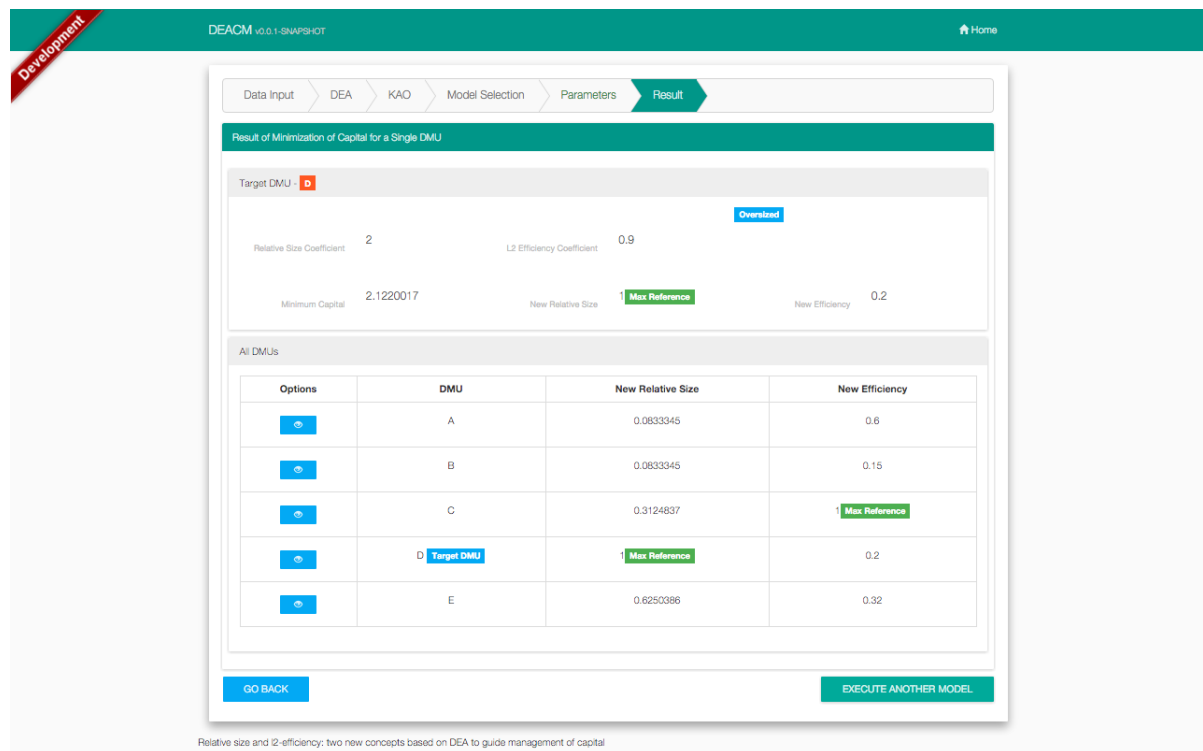


Figure A.15: Result of the Minimization of Capital for a single target DMU

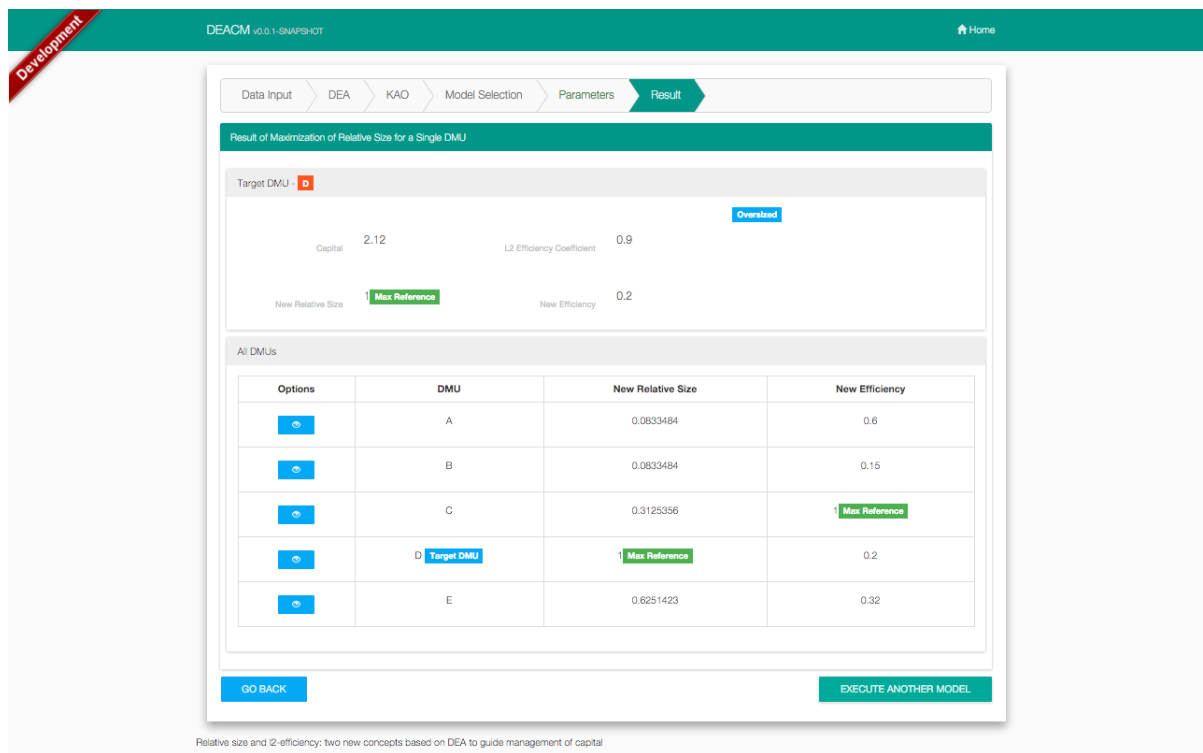


Figure A.16: Result of the Maximization of Relative Size for a single target DMU

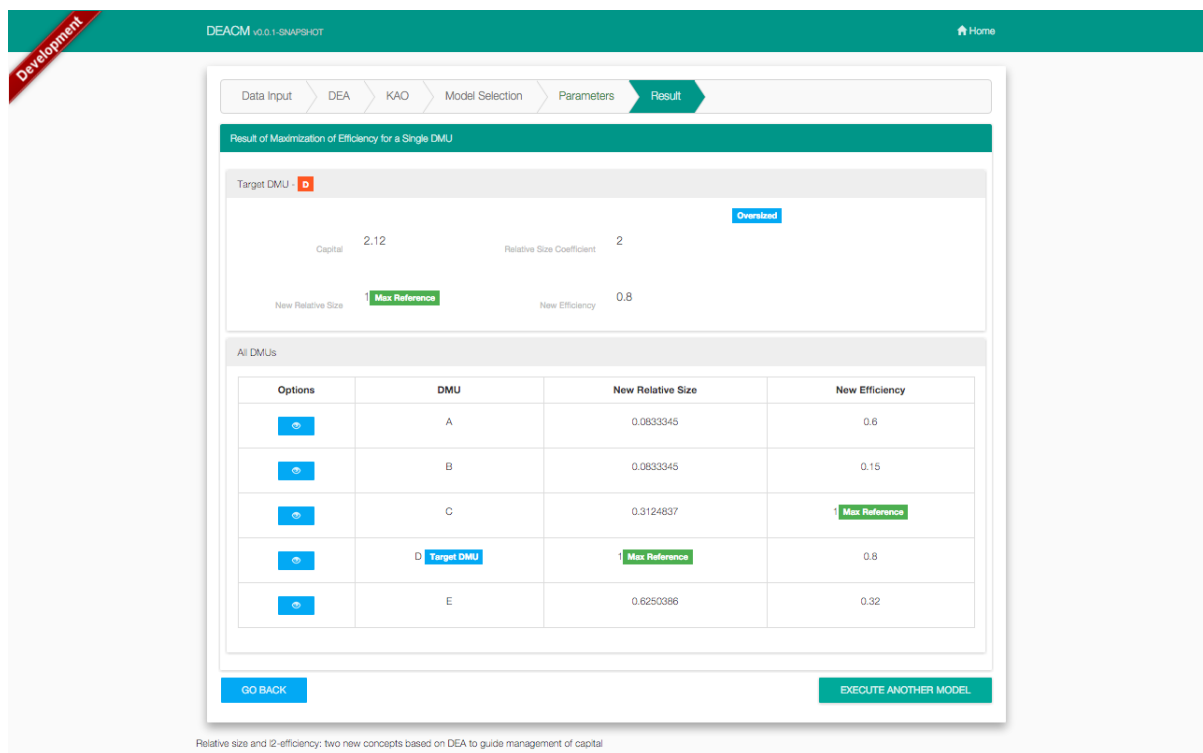


Figure A.17: Result of the Maximization of Relative Efficiency for a single target DMU

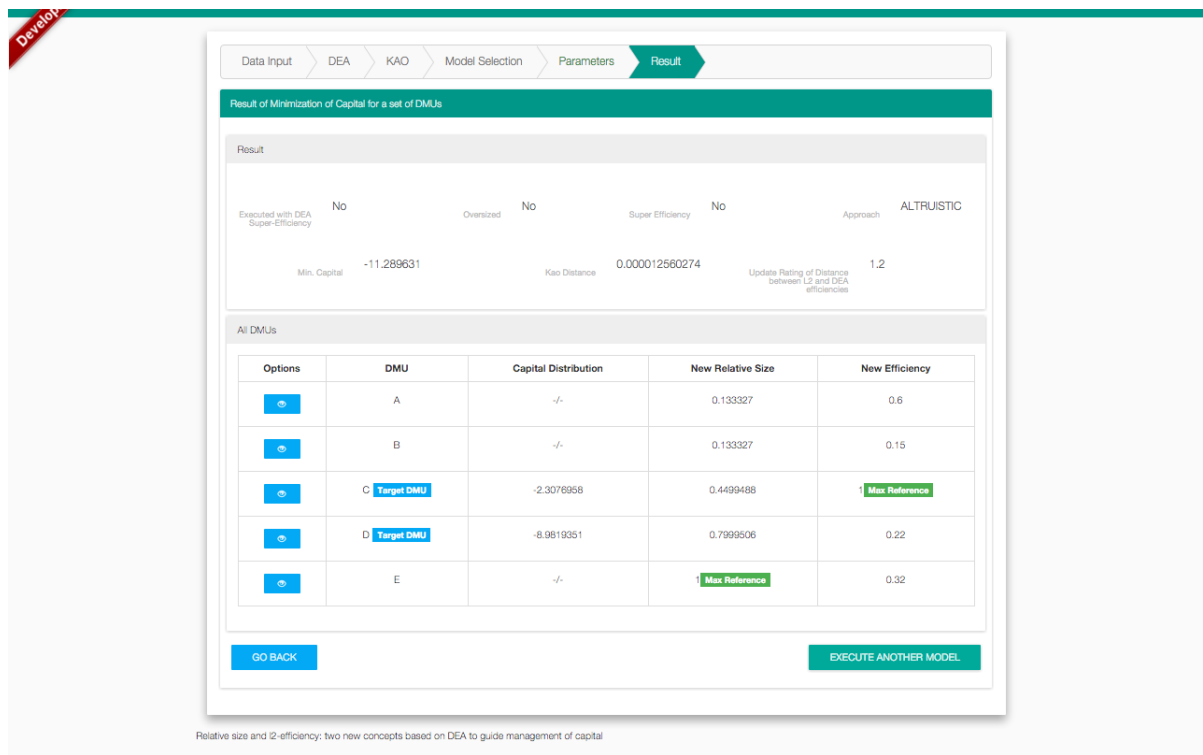


Figure A.18: Result of the Minimization of Capital for *DMUs* subset

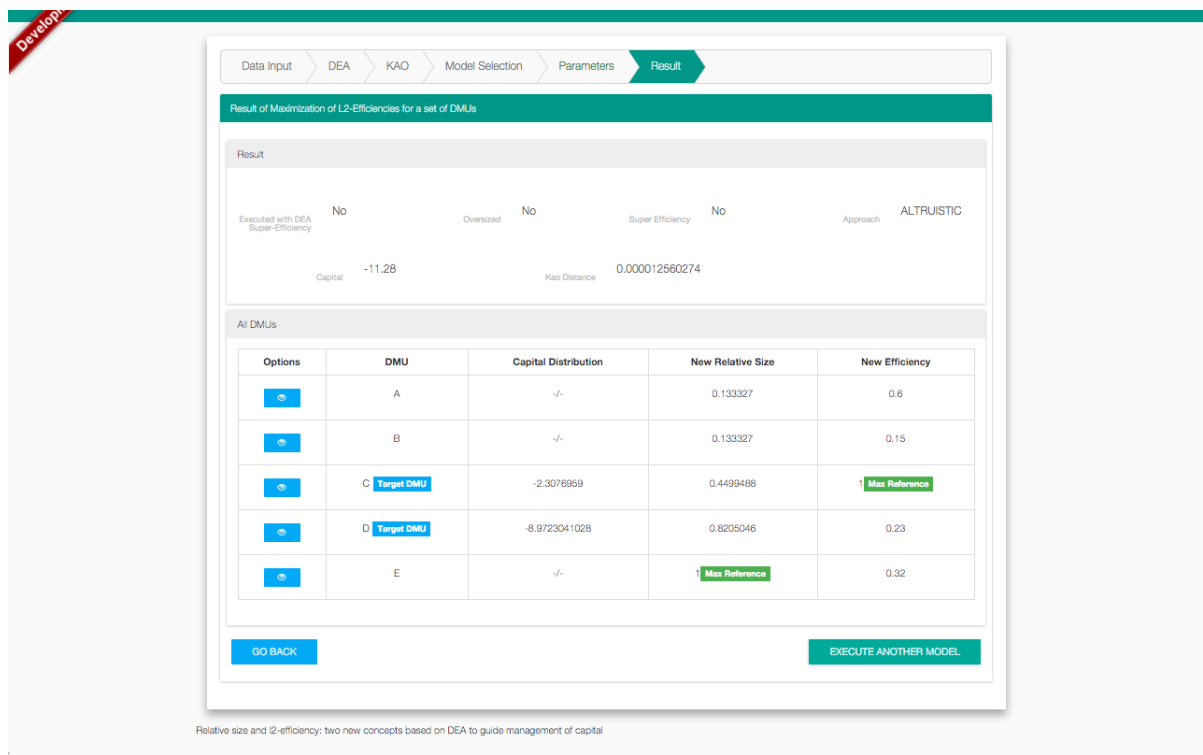


Figure A.19: Result of Maximization of Efficiency for *DMUs* subset

USER'S MANUAL

DEACM

February, 2016